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Massless QCD-like theories

E. M. Cioroianu,* I. Negru,† S. C. Sararu‡
Faculty of Physics, University of Craiova
13 A. I. Cuza Str., Craiova, 200585, Romania

Abstract

Consistent couplings between a set of vector fields and a system of massless Dirac spinors are addressed in the framework of the antifield-BRST deformation procedure.

1 Introduction

In this paper we investigate the problem of the construction of all consistent couplings between a collection of abelian 1-forms and a set of massless Dirac fields in $D = 4$. This problem is solved using the deformation technique of the solution to the classical master equation [1] combined with the local BRST cohomology [2]–[4]. The consistent cross-couplings are derived under the following hypotheses: space-time locality, analyticity of the deformations in the coupling constant, (background) Lorentz invariance, Poincaré invariance (i.e. we do not allow explicit dependence on the spacetime coordinates), preservation of the number of derivatives on each field (such that the differential order of the deformed field equations is preserved with respect to the free model) and the interacting Lagrangian contains at most two space-time derivatives.

2 BRST symmetry of the free model

Our starting point is the free theory that involves a collection of abelian 1-forms and a system of massless Dirac fields in $D = 4$

$$\begin{aligned} S_0^{\mathrm{L}}\left[A_\mu^a, \psi_A, \bar{\psi}_A\right] &= \int d^4x \left[-\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu}\right. \\ &\quad \left. + \frac{\mathrm{i}}{2} \bar{\psi}^A \gamma^\mu \partial_\mu \psi_A - \frac{\mathrm{i}}{2} \left(\partial_\mu \bar{\psi}^A\right) \gamma^\mu \psi_A\right]. \end{aligned} \tag{1}$$

In (1) we used two internal positively-defined metrics k_{ab} and g_{AB} that raise or low the latin indices of the 1-forms and respectively of the Dirac spinors.

Also, we employed the notation $F_{\mu\nu}^a$ for the field-strength of the 1-form A_μ^a ($F_{\mu\nu}^a \equiv \partial_{[\mu} A_{\nu]}^a$).

The free theory possesses the generating set of gauge transformations

$$\delta_\epsilon A_\mu^a = \partial_\mu \epsilon^a, \ \delta_\epsilon \psi_A = \delta_\epsilon \bar{\psi}_A = 0. \tag{2}$$

The gauge parameters ϵ^a are bosonic, and the gauge algebra is Abelian.

The gauge transformations (2) are independent or, in other words, the generating set of gauge transformations is irreducible.

*e-mail address: manache@central.ucv.ro
†e-mail address: inegru@central.ucv.ro
‡e-mail address: scsararu@central.ucv.ro

In order to construct the BRST symmetry for (1) we introduce the field, ghost, and antifield spectra

$$\Phi^{\Gamma_0} = \left(A^a_\mu, \psi_A, \bar{\psi}_A\right), \qquad \Phi^*_{\Gamma_0} = \left(A^{*\mu}_a, \psi^{*A}, \bar{\psi}^{*A}\right) \tag{3}$$

$$\eta^{\Gamma_1} = (\eta^a), \qquad \eta^*_{\Gamma_1} = (\eta^*_a). \tag{4}$$

Since both the gauge generators and the reducibility functions for this model are field-independent, it follows that the BRST differential s reduces to

$$s = \delta + \gamma. \tag{5}$$

The actions of the differentials δ and γ on the generators from the BRST complex are given by

$$\delta A^{*\mu}_a = -\partial_\rho F^{\rho\mu}_a, \quad \delta\psi^{*A} = -\mathrm{i}\partial_\mu\bar{\psi}^A\gamma^\mu, \quad \delta\bar{\psi}^{*A} = -\mathrm{i}\gamma^\mu\partial_\mu\psi^A \tag{6}$$

$$\delta\eta^{*\mu}_a = -\partial_\mu A^{*\mu}_a, \qquad \delta\Phi^{\Gamma_0} = 0, \tag{7}$$

$$\gamma A^a_\mu = \partial_\mu\eta^a \qquad \gamma\psi_A = \gamma\bar{\psi}_A = 0, \qquad \gamma\eta^{\Gamma_1} = 0 \tag{8}$$

$$\gamma\Phi^*_{\Gamma_0} = 0, \qquad \gamma\eta^*_{\Gamma_1} = 0. \tag{9}$$

In this case the anticanonical action of the BRST symmetry, s=(,S), is realized via a solution to the master equation (S,S)=0 that reads as

$$S = S^{\mathrm{L}}_0 + \int d^4x\,(A^{*\mu}_a\partial_\mu\eta^a). \tag{10}$$

3 Construction of consistent interactions

3.1 Cohomological reformulation

We will associate with (10) a deformed solution

$$\begin{aligned} S \quad \rightarrow \quad & \bar{S} = S + \lambda S_1 + \lambda^2 S_2 + \lambda^3 S_3 + \cdots \\ = \quad & S + \lambda\int d^4x\,a + \lambda^2\int d^4x\,b + \lambda^3\int d^4x\,c + \cdots, \end{aligned} \tag{11}$$

which is the BRST generator of the interacting theory, $(\bar{S},\bar{S}) = 0$, such that the components of $\bar{S}$ are restricted to satisfy the tower of equations:

$$(S,S) = 0, \tag{12}$$

$$2\,(S_1,S) = 0, \tag{13}$$

$$2\,(S_2,S) + (S_1,S_1) = 0, \tag{14}$$

$$(S_3,S) + (S_1,S_2) = 0, \tag{15}$$

$$\vdots$$

If we denote by Δ and Λ the nonintegrated densities of the antibrackets (S_1,S_1) and respectively (S_1,S_2) then the local forms of the equations (13)–(15) become

$$sa = \partial_\mu m^\mu, \tag{16}$$

$$2sb + \Delta = \partial_\mu n^\mu, \tag{17}$$

$$sc + \Lambda = \partial_\mu p^\mu, \tag{18}$$

$$\vdots$$

were m^μ, n^μ and p^μ are some local current.

4 Results

4.1 First-order deformation

The non-integrated density of the first-order deformation (the solution to (16)) can be naturally decomposed as

$$a = a^{\mathrm{A}} + a^{\mathrm{int}} + a^{\psi,\bar{\psi}}, \tag{19}$$

each of the terms satisfying independently equations of the type (16).

The piece a^{A} is known from literature [5] its concrete form is

$$a^{\mathrm{A}} = \frac{1}{2} f^{a}_{bc} \left(\eta^{*}_{a} \eta^{b} \eta^{c} - 2A^{*\mu}_{a} \eta^{b} A^{c}_{\mu} - F^{\mu\nu}_{a} A^{b}_{\mu} A^{c}_{\nu} \right), \tag{20}$$

where f^{a}_{bc} are some real constants antisymmetric in its lower indices such that

$$f_{abc} \equiv k_{am} f^{m}_{bc}, \tag{21}$$

are completely antisymmetric.

Theorem 1 *Under the assumptions made in the beginning of the paper supplemented with the reality of the solution to the classical master equation for the deformed theory, the pieces a^{int} and $a^{\psi,\bar{\psi}}$ from (19) [that produce non-trivial deformations] read as*

$$\begin{aligned} a^{\mathrm{int}} \;=\; & \mathrm{i} \left[T_{aAB} \left(\psi^{*A} \psi^{B} - \bar{\psi}^{A} \bar{\psi}^{*B} \right) + \hat{T}_{aAB} \left(\psi^{*A} \gamma_5 \psi^{B} - \bar{\psi}^{A} \gamma_5 \bar{\psi}^{*B} \right) \right] \eta^{a} \\ & + \bar{\psi}^{A} T_{aAB} \gamma^{\mu} \psi^{B} A^{a}_{\mu} + \bar{\psi}^{A} \hat{T}_{aAB} \gamma^{\mu} \gamma_5 \psi^{B} A^{a}_{\mu} \\ & \left(f_{aAB} \bar{\psi}^{A} \gamma^{\mu\nu} \psi^{B} + \hat{f}_{aAB} \bar{\psi}^{A} \gamma^{\mu\nu} \gamma_5 \psi^{B} \right) F^{a}_{\mu\nu} \\ & + \left(f_{abAB} \bar{\psi}^{A} \gamma^{\mu\nu} \gamma^{\alpha\beta} \psi^{B} + \hat{f}_{abAB} \bar{\psi}^{A} \gamma^{\mu\nu} \gamma^{\alpha\beta} \gamma_5 \psi^{B} \right) F^{a}_{\mu\nu} F^{b}_{\alpha\beta}, \end{aligned} \tag{22}$$

$$a^{\psi,\bar{\psi}} = f_0 \left(\psi, \bar{\psi} \right) + f_1 \left(\psi, \bar{\psi}, \partial\psi, \partial\bar{\psi} \right), \tag{23}$$

where:

- T_{aAB} and $\hat{T}_{aAB}$ are the elements of some Hermitic matrices;

- f_{aAB}, $\hat{f}_{aAB}$, f_{abAB} and $\hat{f}_{abAB}$ are functions that depends only on the undifferentiated Dirac spinors that satisfy the following algebraic properties

$$\left(f_{aAB} \right)^{\star} = -f_{aBA}, \qquad \left(\hat{f}_{aAB} \right)^{\star} = \hat{f}_{aBA}, \tag{24}$$

$$\left(f_{abAB} \right)^{\star} = f_{baBA}, \qquad \left(\hat{f}_{abAB} \right)^{\star} = -\hat{f}_{baBA}; \tag{25}$$

- $f_0 \left(\psi, \bar{\psi} \right)$ is an arbitrary real function that depend only on the undifferentiated spinors and $f_1 \left(\psi, \bar{\psi}, \partial\psi, \partial\bar{\psi} \right)$ is a real function that depends on the massless Dirac fields and contains only one space-time derivative.

Inserting (20), (22) and (23) in (19) we derive the most general and non-trivial solution to the equation (16), solution that represents the non-integrated density of the first-order deformation.

On behalf of the basis

$$\left\{ \mathbf{1},\ \gamma_{\mu},\ \gamma_{\mu\nu},\ \gamma_{\mu}\gamma_5,\ \gamma_5 \right\}, \tag{26}$$

in the space of 4×4 complex matrices, the real function $f_1 \left(\psi, \bar{\psi}, \partial\psi, \partial\bar{\psi} \right)$ can be written as

$$\begin{aligned} f_1 \;=\; & \left(m^{\mu} \right)_{AB} \bar{\psi}^{A} \partial_{\mu} \psi^{B} + \left[\left(m^{\mu} \right)_{BA} \right]^{\star} \left(\partial_{\mu} \bar{\psi}^{A} \right) \psi^{B} + \left(m^{\mu}_{\alpha} \right)_{AB} \bar{\psi}^{A} \gamma^{\alpha} \partial_{\mu} \psi^{B} \\ & + \left[\left(m^{\mu}_{\alpha} \right)_{BA} \right]^{\star} \left(\partial_{\mu} \bar{\psi}^{A} \right) \gamma^{\alpha} \psi^{B} + \left(m^{\mu}_{\alpha\beta} \right)_{AB} \bar{\psi}^{A} \gamma^{\alpha\beta} \partial_{\mu} \psi^{B} \\ & - \left[\left(m^{\mu}_{\alpha\beta} \right)_{BA} \right]^{\star} \left(\partial_{\mu} \bar{\psi}^{A} \right) \gamma^{\alpha\beta} \psi^{B} + \left(\hat{m}^{\mu}_{\alpha} \right)_{AB} \bar{\psi}^{A} \gamma^{\alpha} \gamma_5 \partial_{\mu} \psi^{B} \\ & + \left[\left(\hat{m}^{\mu}_{\alpha} \right)_{BA} \right]^{\star} \left(\partial_{\mu} \bar{\psi}^{A} \right) \gamma^{\alpha} \gamma_5 \psi^{B} \\ & + \left(\hat{m}^{\mu} \right)_{AB} \bar{\psi}^{A} \gamma_5 \partial_{\mu} \psi^{B} - \left[\left(\hat{m}^{\mu} \right)_{BA} \right]^{\star} \left(\partial_{\mu} \bar{\psi}^{A} \right) \gamma_5 \psi^{B}, \end{aligned} \tag{27}$$

where all the quantities $(m^{\mu}_{\Delta})_{AB}$ and $(\hat{m}^{\mu}_{\Delta})_{AB}$ are arbitrary functions that depend only on the undifferentiated Dirac fields.

4.2 Higher-order deformations

The next equation that we have to solve is (14) with its local expression (17).

The non-integrated density of the second-order deformation admits a similar decomposition as the first-order one

$$b = b^{\mathrm{A}} + b^{\mathrm{int}} + b^{\psi,\bar{\psi}}, \tag{28}$$

each of the pieces satisfies independently equations of the form (17)

$$\begin{aligned} 2sb^{\mathrm{A}} + \Delta^{\mathrm{A}} &= \partial^{\mu} n^{\mathrm{A}}_{\mu}, && (29)\\ 2sb^{\psi,\bar{\psi}} + \Delta^{\psi,\bar{\psi}} &= \partial^{\mu} n^{\psi,\bar{\psi}}_{\mu}, && (30)\\ 2sb^{\mathrm{int}} + \Delta^{\mathrm{int}} &= \partial^{\mu} n^{\mathrm{int}}_{\mu}. && (31) \end{aligned}$$

The resolution to (29) is already known [5].

Precisely, the existence of b^{A} requires for the real constants f^{a}_{bc} to satisfy

$$f^{m}_{[ab} f^{d}_{c]m} = 0. \tag{32}$$

With the identity (32) at hand, the solution to (29) reads as

$$b^{\mathrm{A}} = -\frac{1}{4} f_{mab} f^{m}_{cd} A^{a\mu} A^{b\nu} A^{c}_{\mu} A^{d}_{\nu}. \tag{33}$$

By direct computation it can be shown that

$$\Delta^{\psi,\bar{\psi}} = 0, \tag{34}$$

such that we can take

$$b^{\psi,\bar{\psi}} = 0. \tag{35}$$

The equation (31) can be solved by projecting it on various antighost numbers and on various numbers of derivatives.

The resolution to the equation (31) can be summarized in the following theorem:

Theorem 2 *i) The existence of the b^{int} as solution for (31) requires for the functions [constants] that parametrize the first-order deformation to satisfy the equations*

$$\left[T_a, \hat{T}_b\right] + \left[\hat{T}_a, T_b\right] = i f^{m}_{ab} \hat{T}_m, \quad [T_a, T_b] + \left[\hat{T}_a, \hat{T}_b\right] = \mathrm{i} f^{m}_{ab} T_m, \tag{36}$$

$$\begin{aligned} &T_{aAB} \left(\frac{\partial^{\mathrm{R}} f_0}{\partial \psi_A} \psi^B - \bar{\psi}^A \frac{\partial^{\mathrm{L}} f_0}{\partial \bar{\psi}_B} \right) \\ &+ \hat{T}_{aAB} \left(\frac{\partial^{\mathrm{R}} f_0}{\partial \psi_A} \gamma_5 \psi^B + \bar{\psi}^A \gamma_5 \frac{\partial^{\mathrm{L}} f_0}{\partial \bar{\psi}_B} \right) = 0, \end{aligned} \tag{37}$$

$$\begin{aligned} &[f_a, T_b]_{AB} + \left\{\hat{f}_a, \hat{T}_b\right\}_{AB} + T_{bCD} \left(\frac{\partial^{\mathrm{R}} f_{aAB}}{\partial \psi_C} \psi^D - \bar{\psi}^C \frac{\partial^{\mathrm{L}} f_{aAB}}{\partial \bar{\psi}_D} \right) \\ &+ \hat{T}_{bCD} \left(\frac{\partial^{\mathrm{R}} f_{aAB}}{\partial \psi_C} \gamma_5 \psi^D + \bar{\psi}^C \gamma_5 \frac{\partial^{\mathrm{L}} f_{aAB}}{\partial \bar{\psi}_D} \right) - \mathrm{i} f^{m}_{ab} f_{mAB} = 0, \end{aligned} \tag{38}$$

$$\left[\hat{f}_a, T_b\right]_{AB} + \left\{f_a, \hat{T}_b\right\}_{AB} + T_{bCD}\left(\frac{\partial^{\mathrm{R}}\hat{f}_{aAB}}{\partial\psi_C}\psi^D - \bar{\psi}^C\frac{\partial^{\mathrm{L}}\hat{f}_{aAB}}{\partial\bar{\psi}_D}\right)$$
$$+\hat{T}_{bCD}\left(\frac{\partial^{\mathrm{R}}\hat{f}_{aAB}}{\partial\psi_C}\gamma_5\psi^D + \bar{\psi}^C\gamma_5\frac{\partial^{\mathrm{L}}\hat{f}_{aAB}}{\partial\bar{\psi}_D}\right) - \mathrm{i}f_{ab}^{m}\hat{f}_{mAB} = 0, \tag{39}$$

$$\left[m^\mu, T_b\right]_{AB} + \left\{\hat{m}^\mu, \hat{T}_b\right\}_{AB} + T_{bCD}\left(\frac{\partial^{\mathrm{R}}\left(m^\mu\right)_{AB}}{\partial\psi_C}\psi^D - \bar{\psi}^C\frac{\partial^{\mathrm{L}}\left(m^\mu\right)_{AB}}{\partial\bar{\psi}_D}\right)$$
$$+\hat{T}_{bCD}\left(\frac{\partial^{\mathrm{R}}\left(m^\mu\right)_{AB}}{\partial\psi_C}\gamma_5\psi^D + \bar{\psi}^C\gamma_5\frac{\partial^{\mathrm{L}}\left(m^\mu\right)_{AB}}{\partial\bar{\psi}_D}\right) = 0, \tag{40}$$

$$\left[m_\alpha^\mu, T_b\right]_{AB} + \left[\hat{m}_\alpha^\mu, \hat{T}_b\right]_{AB} + T_{bCD}\left(\frac{\partial^{\mathrm{R}}\left(m_\alpha^\mu\right)_{AB}}{\partial\psi_C}\psi^D - \bar{\psi}^C\frac{\partial^{\mathrm{L}}\left(m_\alpha^\mu\right)_{AB}}{\partial\bar{\psi}_D}\right)$$
$$+\hat{T}_{bCD}\left(\frac{\partial^{\mathrm{R}}\left(m_\alpha^\mu\right)_{AB}}{\partial\psi_C}\gamma_5\psi^D + \bar{\psi}^C\gamma_5\frac{\partial^{\mathrm{L}}\left(m_\alpha^\mu\right)_{AB}}{\partial\bar{\psi}_D}\right) = 0, \tag{41}$$

$$\left[m_{\alpha\beta}^\mu, T_b\right]_{AB} + \frac{\mathrm{i}}{2}\varepsilon_{\alpha\beta}{}^{\gamma\delta}\left\{m_{\gamma\delta}^\mu, \hat{T}_b\right\}_{AB}$$
$$+T_{bCD}\left(\frac{\partial^{\mathrm{R}}\left(m_{\alpha\beta}^\mu\right)_{AB}}{\partial\psi_C}\psi^D - \bar{\psi}^C\frac{\partial^{\mathrm{L}}\left(m_{\alpha\beta}^\mu\right)_{AB}}{\partial\bar{\psi}_D}\right)$$
$$+\hat{T}_{bCD}\left(\frac{\partial^{\mathrm{R}}\left(m_{\alpha\beta}^\mu\right)_{AB}}{\partial\psi_C}\gamma_5\psi^D + \bar{\psi}^C\gamma_5\frac{\partial^{\mathrm{L}}\left(m_{\alpha\beta}^\mu\right)_{AB}}{\partial\bar{\psi}_D}\right) = 0, \tag{42}$$

$$\left[\hat{m}_\alpha^\mu, T_b\right]_{AB} + \left[m_\alpha^\mu, \hat{T}_b\right]_{AB} + T_{bCD}\left(\frac{\partial^{\mathrm{R}}\left(\hat{m}_\alpha^\mu\right)_{AB}}{\partial\psi_C}\psi^D - \bar{\psi}^C\frac{\partial^{\mathrm{L}}\left(\hat{m}_\alpha^\mu\right)_{AB}}{\partial\bar{\psi}_D}\right)$$
$$+\hat{T}_{bCD}\left(\frac{\partial^{\mathrm{R}}\left(\hat{m}_\alpha^\mu\right)_{AB}}{\partial\psi_C}\gamma_5\psi^D + \bar{\psi}^C\gamma_5\frac{\partial^{\mathrm{L}}\left(\hat{m}_\alpha^\mu\right)_{AB}}{\partial\bar{\psi}_D}\right) = 0, \tag{43}$$

$$\left[\hat{m}^\mu, T_b\right]_{AB} + \left\{m^\mu, \hat{T}_b\right\}_{AB} + T_{bCD}\left(\frac{\partial^{\mathrm{R}}\left(\hat{m}^\mu\right)_{AB}}{\partial\psi_C}\psi^D - \bar{\psi}^C\frac{\partial^{\mathrm{L}}\left(\hat{m}^\mu\right)_{AB}}{\partial\bar{\psi}_D}\right)$$
$$+\hat{T}_{bCD}\left(\frac{\partial^{\mathrm{R}}\left(\hat{m}^\mu\right)_{AB}}{\partial\psi_C}\gamma_5\psi^D + \bar{\psi}^C\gamma_5\frac{\partial^{\mathrm{L}}\left(\hat{m}^\mu\right)_{AB}}{\partial\bar{\psi}_D}\right) = 0, \tag{44}$$

$$\left[f_{ab}, T_c\right]_{AB} + \left\{\hat{f}_{ab}, \hat{T}_c\right\}_{AB} + T_{cCD}\left(\frac{\partial^{\mathrm{R}}f_{abAB}}{\partial\psi_C}\psi^D - \bar{\psi}^C\frac{\partial^{\mathrm{L}}f_{abAB}}{\partial\bar{\psi}_D}\right)$$
$$+\hat{T}_{cCD}\left(\frac{\partial^{\mathrm{R}}f_{abAB}}{\partial\psi_C}\gamma_5\psi^D + \bar{\psi}^C\gamma_5\frac{\partial^{\mathrm{L}}f_{abAB}}{\partial\bar{\psi}_D}\right)$$
$$-\mathrm{i}\left(f_{bc}^{m}f_{amAB} + f_{ac}^{m}f_{mbAB}\right) = 0, \tag{45}$$

$$\left[\hat{f}_{ab}, T_c\right]_{AB} + \left\{f_{ab}, \hat{T}_c\right\}_{AB} + T_{cCD}\left(\frac{\partial^{\mathrm{R}}\hat{f}_{abAB}}{\partial\psi_C}\psi^D - \bar{\psi}^C\frac{\partial^{\mathrm{L}}\hat{f}_{abAB}}{\partial\bar{\psi}_D}\right)$$
$$+\hat{T}_{cCD}\left(\frac{\partial^{\mathrm{R}}\hat{f}_{abAB}}{\partial\psi_C}\gamma_5\psi^D + \bar{\psi}^C\gamma_5\frac{\partial^{\mathrm{L}}\hat{f}_{abAB}}{\partial\bar{\psi}_D}\right)$$
$$-\mathrm{i}\left(f_{bc}^{m}\hat{f}_{amAB} + f_{ac}^{m}\hat{f}_{mbAB}\right) = 0; \tag{46}$$

ii) The concrete expression of the non-integrated density of the second-order deformation in the interacting sector reads as

$$
\begin{aligned}
b^{\text{int}} = {} & f^{a}_{bc}A^{b}_{\mu}A^{c}_{\nu}\left(f_{aAB}\bar{\psi}^{A}\gamma^{\mu\nu}\psi^{B}+\hat{f}_{aAB}\bar{\psi}^{A}\gamma^{\mu\nu}\psi^{B}\right)-\mathrm{i}\bar{\psi}^{A}\psi^{B}A^{a}_{\mu}\left[m^{\mu}T_{a}\right.\\
& \left.+\hat{m}^{\mu}\hat{T}_{a}-T_{a}\left(m^{\mu}\right)^{\dagger}-\hat{T}_{a}\left(\hat{m}^{\mu}\right)^{\dagger}\right]_{AB}-\mathrm{i}\bar{\psi}^{A}\gamma^{\alpha}\psi^{B}A^{a}_{\mu}\left[m^{\mu}_{\alpha}T_{a}+\hat{m}^{\mu}_{\alpha}\hat{T}_{a}\right.\\
& \left.-T_{a}\left(m^{\mu}_{\alpha}\right)^{\dagger}-\hat{T}_{a}\left(\hat{m}^{\mu}_{\alpha}\right)^{\dagger}\right]_{AB}-\mathrm{i}\bar{\psi}^{A}\gamma^{\alpha\beta}\psi^{B}A^{a}_{\mu}\left[m^{\mu}_{\alpha\beta}T_{a}+T_{a}\left(m^{\mu}_{\alpha\beta}\right)^{\dagger}\right]_{AB}\\
& -\mathrm{i}\bar{\psi}^{A}\gamma^{\alpha\beta}\gamma_{5}\psi^{B}A^{a}_{\mu}\left[m^{\mu}_{\alpha\beta}\hat{T}_{a}-\hat{T}_{a}\left(m^{\mu}_{\alpha\beta}\right)^{\dagger}\right]_{AB}\\
& -\mathrm{i}\bar{\psi}^{A}\gamma^{\alpha}\gamma_{5}\psi^{B}A^{a}_{\mu}\left[\hat{m}^{\mu}_{\alpha}T_{a}+m^{\mu}_{\alpha}\hat{T}_{a}-T_{a}\left(\hat{m}^{\mu}_{\alpha}\right)^{\dagger}-\hat{T}_{a}\left(m^{\mu}_{\alpha}\right)^{\dagger}\right]_{AB}\\
& -\mathrm{i}\bar{\psi}^{A}\gamma_{5}\psi^{B}A^{a}_{\mu}\left[\hat{m}^{\mu}T_{a}+m^{\mu}\hat{T}_{a}-T_{a}\left(\hat{m}^{\mu}\right)^{\dagger}-\hat{T}_{a}\left(m^{\mu}\right)^{\dagger}\right]_{AB}\\
& +f^{b}_{mn}F^{a}_{\mu\nu}A^{m}_{\alpha}A^{n}_{\beta}\left(f_{abAB}\bar{\psi}^{A}\gamma^{\mu\nu}\gamma^{\alpha\beta}\psi^{B}+\hat{f}_{abAB}\bar{\psi}^{A}\gamma^{\mu\nu}\gamma^{\alpha\beta}\gamma_{5}\psi^{B}\right)\\
& +f^{a}_{mn}F^{b}_{\alpha\beta}A^{m}_{\mu}A^{n}_{\nu}\left(f_{abAB}\bar{\psi}^{A}\gamma^{\mu\nu}\gamma^{\alpha\beta}\psi^{B}+\hat{f}_{abAB}\bar{\psi}^{A}\gamma^{\mu\nu}\gamma^{\alpha\beta}\gamma_{5}\psi^{B}\right).
\end{aligned}
\tag{47}
$$

The non-integrated density of the third-order deformation can be decomposed like the first- and second-order ones

$$c=c^{\mathrm{A}}+c^{\mathrm{int}}+c^{\psi,\bar{\psi}}. \tag{48}$$

These terms satisfy equations similar to (18)

$$sc^{\mathrm{A}}+\Lambda^{\mathrm{A}} = \partial^{\mu}p^{\mathrm{A}}_{\mu}, \tag{49}$$

$$sc^{\psi,\bar{\psi}}+\Lambda^{\psi,\bar{\psi}} = \partial^{\mu}p^{\psi,\bar{\psi}}_{\mu}, \tag{50}$$

$$sc^{\mathrm{int}}+\Lambda^{\mathrm{int}} = \partial^{\mu}p^{\mathrm{int}}_{\mu}. \tag{51}$$

It was shown [5] that the solution to (49) can be taken

$$c^{\mathrm{A}}=0, \tag{52}$$

because on the identity (32)

$$\Lambda^{\mathrm{A}}=0. \tag{53}$$

The equation (50) has also trivial solution on behalf of

$$\Lambda^{\psi,\bar{\psi}}=0. \tag{54}$$

By direct computations, the solution of the equation (51) on the identities (45) and (46) reads as

$$c^{\mathrm{int}}=f^{a}_{mn}f^{b}_{pq}A^{m}_{\mu}A^{n}_{\nu}A^{p}_{\alpha}A^{q}_{\beta}\left(f_{abAB}\bar{\psi}^{A}\gamma^{\mu\nu}\gamma^{\alpha\beta}\psi^{B}+\hat{f}_{abAB}\bar{\psi}^{A}\gamma^{\mu\nu}\gamma^{\alpha\beta}\gamma_{5}\psi^{B}\right). \tag{55}$$

The consistency of the third-order deformation of the solution to the classical master equation, on the identities (45) and (46) leads to the fact that fourth-order deformation can be taken zero

$$S_{4}=0, \tag{56}$$

so the deformation procedure stops at the third-order in the coupling constant.

5 The interacting theory

Putting the results obtained in the above together, we can write the solution of the classical master equation for the interacting theory

$$\bar{S}=S+\int d^{4}x\left(\lambda a+\lambda^{2}b+\lambda^{3}c\right), \tag{57}$$

where the terms displayed in (57) are explicitly given in (10), (19), (28) and (48) and in addition, the functions (constants) that parametrize (57) satisfy the equations (32) and (63)–(46).

From (57) we can identify the Lagrangian formulation of the interacting theory as follows:

-the antighost number zero pieces from (57) give us the Lagrangian action of the interacting theory

$$\begin{aligned}
&\bar{S}_0^{\mathrm{L}}\left[A_\mu^a,\psi_A,\bar{\psi}_A\right]=\int d^4x\left\{-\frac{1}{4}\bar{F}^a_{\mu\nu}\bar{F}_a^{\mu\nu}+\frac{\mathrm{i}}{2}\bar{\psi}^A\gamma^\mu\left(k_{AB}\overrightarrow{\partial}_\mu-\mathrm{i}\lambda A_\mu^a T_{aAB}\right.\right.\\
&\left.-\mathrm{i}\lambda A_\mu^a\hat{T}_{aAB}\gamma_5\right)\psi^B-\frac{\mathrm{i}}{2}\bar{\psi}^A\left(k_{AB}\overleftarrow{\partial}_\mu+\mathrm{i}\lambda A_\mu^a T_{aAB}-\mathrm{i}\lambda A_\mu^a\hat{T}_{aAB}\gamma_5\right)\gamma^\mu\psi^B\\
&+\lambda f_0\left(\psi,\bar{\psi}\right)+\lambda\left[\left(m^\mu\right)_{AB}\bar{\psi}^A\partial_\mu\psi^B+\left[\left(m^\mu\right)_{BA}\right]^\star\left(\partial_\mu\bar{\psi}^A\right)\psi^B\right.\\
&+\left(m^\mu_\alpha\right)_{AB}\bar{\psi}^A\gamma^\alpha\partial_\mu\psi^B+\left[\left(m^\mu_\alpha\right)_{BA}\right]^\star\left(\partial_\mu\bar{\psi}^A\right)\gamma^\alpha\psi^B\\
&+\left(m^\mu_{\alpha\beta}\right)_{AB}\bar{\psi}^A\gamma^{\alpha\beta}\partial_\mu\psi^B-\left[\left(m^\mu_{\alpha\beta}\right)_{BA}\right]^\star\left(\partial_\mu\bar{\psi}^A\right)\gamma^{\alpha\beta}\psi^B\\
&+\left(\hat{m}^\mu_\alpha\right)_{AB}\bar{\psi}^A\gamma^\alpha\gamma_5\partial_\mu\psi^B+\left[\left(\hat{m}^\mu_\alpha\right)_{BA}\right]^\star\left(\partial_\mu\bar{\psi}^A\right)\gamma^\alpha\gamma_5\psi^B\\
&\left.+\left(\hat{m}^\mu\right)_{AB}\bar{\psi}^A\gamma_5\partial_\mu\psi^B-\left[\left(\hat{m}^\mu\right)_{BA}\right]^\star\left(\partial_\mu\bar{\psi}^A\right)\gamma_5\psi^B\right]\\
&-\mathrm{i}\lambda^2\bar{\psi}^A\psi^B A_\mu^a\left[m^\mu T_a+\hat{m}^\mu\hat{T}_a-T_a\left(m^\mu\right)^\dagger-\hat{T}_a\left(\hat{m}^\mu\right)^\dagger\right]_{AB}\\
&-\mathrm{i}\lambda^2\bar{\psi}^A\gamma^\alpha\psi^B A_\mu^a\left[m^\mu_\alpha T_a+\hat{m}^\mu_\alpha\hat{T}_a-T_a\left(m^\mu_\alpha\right)^\dagger-\hat{T}_a\left(\hat{m}^\mu_\alpha\right)^\dagger\right]_{AB}\\
&-\mathrm{i}\lambda^2\bar{\psi}^A\gamma^{\alpha\beta}\psi^B A_\mu^a\left[m^\mu_{\alpha\beta}T_a+T_a\left(m^\mu_{\alpha\beta}\right)^\dagger\right]_{AB}\\
&-\mathrm{i}\lambda^2\bar{\psi}^A\gamma^{\alpha\beta}\gamma_5\psi^B A_\mu^a\left[m^\mu_{\alpha\beta}\hat{T}_a-\hat{T}_a\left(m^\mu_{\alpha\beta}\right)^\dagger\right]_{AB}\\
&-\mathrm{i}\lambda^2\bar{\psi}^A\gamma^\alpha\gamma_5\psi^B A_\mu^a\left[\hat{m}^\mu_\alpha T_a+m^\mu_\alpha\hat{T}_a-T_a\left(\hat{m}^\mu_\alpha\right)^\dagger-\hat{T}_a\left(m^\mu_\alpha\right)^\dagger\right]_{AB}\\
&-\mathrm{i}\lambda^2\bar{\psi}^A\gamma_5\psi^B A_\mu^a\left[\hat{m}^\mu T_a+m^\mu\hat{T}_a-T_a\left(\hat{m}^\mu\right)^\dagger-\hat{T}_a\left(m^\mu\right)^\dagger\right]_{AB}\\
&+\lambda\left(f_{aAB}\bar{\psi}^A\gamma^{\mu\nu}\psi^B+\hat{f}_{aAB}\bar{\psi}^A\gamma^{\mu\nu}\gamma_5\psi^B\right)\bar{F}^a_{\mu\nu}\\
&\left.+\lambda\left(f_{abAB}\bar{\psi}^A\gamma^{\mu\nu}\gamma^{\alpha\beta}\psi^B+\hat{f}_{abAB}\bar{\psi}^A\gamma^{\mu\nu}\gamma^{\alpha\beta}\gamma_5\psi^B\right)\bar{F}^a_{\mu\nu}\bar{F}^b_{\alpha\beta}\right\},
\end{aligned}\tag{58}$$

where we used the notation

$$\bar{F}^a_{\mu\nu}=\partial_{[\mu}A^a_{\nu]}+\lambda f^a_{bc}A^b_\mu A^c_\nu.\tag{59}$$

-the antighost number one pieces from (57) offer us the generating set of gauge transformations for the deformed model

$$\bar{\delta}_\epsilon A_\mu^a=\partial_\mu\epsilon^a+\lambda f^a_{bc}A^b_\mu\epsilon^c,\tag{60}$$

$$\bar{\delta}_\epsilon\psi_A=\mathrm{i}\lambda\left(T_{aAB}+\gamma_5\hat{T}_{aAB}\right)\psi^B\epsilon^a,\tag{61}$$

$$\bar{\delta}_\epsilon\bar{\psi}_A=-\mathrm{i}\lambda\bar{\psi}^B\left(T_{aBA}-\gamma_5\hat{T}_{aBA}\right)\epsilon^a.\tag{62}$$

6 Algebraic interpretation of the generators of deformed gauge transformations

Let's analyze the equations

$$\left[T_a,\hat{T}_b\right]+\left[\hat{T}_a,T_b\right]=if^m_{ab}\hat{T}_m,\quad\left[T_a,T_b\right]+\left[\hat{T}_a,\hat{T}_b\right]=\mathrm{i}f^m_{ab}T_m,\tag{63}$$

satisfied by the generators T_a and $\hat{T}_a$ of the deformed gauge transformations.

We define the Hermitic matrices

$$\mathcal{T}_a=T_a+\hat{T}_a,\qquad\mathcal{F}_a=T_a-\hat{T}_a.\tag{64}$$

Using the equations (63), the Hermitic matrices defined in the above satisfy

$$[\mathcal{T}_a, \mathcal{T}_b] = \mathrm{i} f_{ab}^{m} \mathcal{T}_m, \qquad [\mathcal{F}_a, \mathcal{F}_b] = \mathrm{i} f_{ab}^{m} \mathcal{F}_m. \tag{65}$$

From (65) we can interpret $(\mathcal{T}_a)_a$ and $(\mathcal{F}_a)_a$ as the generators of two inequivalent unitary representations of the Lie group those Lie algebra has the structure constants f_{ab}^{m}.

7 Conclusion

To conclude with, in this paper we have investigated the couplings between a collection of abelian 1-forms and a set of massless Dirac fields in $D = 4$ using the powerful setting based on local BRST cohomology. The interacting theory is parametrized by ten sets of smooth functions of the undifferentiated spinors and three systems of constants which satisfy twelve equations. The gauge algebra of the deformed theory is a Lie algebra that closes off-shell and the interacting Lagrangian is of order three in the coupling constant.

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