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Quantum Fields on Noncommutative Spaces

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Abstract

We review the operatorial quantization of fields defined on noncommutative spaces, offering an insightful alternative to the usual (Wigner-Weyl-)Moyal procedure. One can demonstrate that the elementary degrees of freedom are bilocal, and live on a reduced configuration space. Perturbation theory is easily formulated, and the IR-UV mixing is simply interpreted in this framework. Causality issues become particularly transparent.

Noncommutative (NC) field theories (FT) [1] attracted considerable attention over the last decade. Heuristic arguments appeared to favour a dipolar nature of the degrees of freedom of such theories [2, 3]. Nevertheless, the terminology employed is more vague, referring to nonlocality, or to "nonlocality of a peculiar nature". Also, most of the work done in the field uses the Weyl-Moyal approach.

We review here a different approach, based on a canonical quantization procedure [4], which simply but rigourously demonstrates the intrinsic *bilocal* nature of noncommutative fields. By avoiding reference to the Weyl space of symbols (implicit in the Moyal approach) it renders transparent the nature of the real space-time on which dynamics takes place, and on which measurements could be performed. This approach allows one to view our space from different perspectives [4, 5], corresponding to the representation of the NC algebra one chooses. It also permits to show the finite character of classical noncommutative waves, even at the location of the source [5]. Interactions are naturally described in terms of dipoles, and an instructive interpretation of the IR/UV mixing becomes available. Causality is unambiguously defined and shown to hold at least for theories with commutative time.

Bilocal objects

The simplest NC field is a $(2+1)$-dimensional scalar $\Phi(t, \hat{x}, \hat{y})$, defined over a commuting time t and a pair of NC coordinates which satisfy

$$[\hat{x}, \hat{y}] = i\theta. \tag{1}$$

The extension to several NC pairs is straightforward. The action is

$$S = \frac{1}{2}\int \mathrm{d}t\, \mathrm{Tr}_{\mathcal{H}} \left[\dot{\Phi}^2 - (\partial_x\Phi)^2 - (\partial_y\Phi)^2 - m^2\Phi^2 - 2V(\Phi)\right]. \tag{2}$$

We will exemplify with a quartic potential, $V(\Phi) = \frac{g}{4!}\Phi^4$. The operators $\hat{x}$ and $\hat{y}$ act on a harmonic oscillator Hilbert space $\mathcal{H}$ in the usual way. $\mathcal{H}$ may be given a discrete basis $\{|n>\}$ formed by eigenstates of $\hat{x}^2 + \hat{y}^2$ [5], or a continuous one $\{|x>\}$, composed of eigenstates of, say, $\hat{x}$ [4]. Another possibility is to use as a basis the standard coherent states.

To quantize Φ [4], we start with a usual classical commuting field, expanded into normal modes with coefficients a and a^*. Upon usual field quantization, a and a^* become operators acting on a standard Fock space $\mathcal{F}$. To make the underlying space noncommutative, let us introduce (1) and apply the Weyl (not Weyl-Moyal!) quantization procedure to the exponentials $e^{i(k_x x + k_y y)}$. The result is

$$\Phi = \int\int \frac{dk_x dk_y}{2\pi\sqrt{2\omega_{\vec{k}}}} \left[\hat{a}_{k_x k_y} e^{i(\omega_{\vec{k}} t - k_x\hat{x} - k_y\hat{y})} + \hat{a}^\dagger_{k_x k_y} e^{-i(\omega_{\vec{k}} t - k_x\hat{x} - k_y\hat{y})}\right]. \tag{3}$$

which means the following: Φ is a 'doubly'-quantum field operator, acting on a direct product of two Hilbert spaces, $\Phi : \mathcal{F} \otimes \mathcal{H} \to \mathcal{F} \otimes \mathcal{H}$. Physically, Φ creates (destroys), via $\hat{a}^{\dagger}_{k_x k_y}$ ($\hat{a}_{k_x k_y}$), an excitation represented by an "operatorial plane wave" $e^{i(\omega_{\vec{k}} t - k_x \hat{x} - k_y \hat{y})}$. We now discuss the nature of such an excitation.

One could work with Φ as an operator ready to act on both $\mathcal{F}$ and $\mathcal{H}$. It is however simpler to saturate its action on $\mathcal{H}$, working with expectation values $< x'|\Phi|x >: \mathcal{F} \to \mathcal{F}$. It is at this point, while eliminating noncommutativity, that bilocality appears. To see that, consider the family $\{|x>\}$ of eigenstates of $\hat{x}$: $\hat{x}|x>= x|x>$, $\hat{y}|x>= -i\theta \frac{\partial}{\partial x}|x>$. A simple but key equation is

$$< x'|e^{i(k_x\hat{x}+k_y\hat{y})}|x>= e^{ik_x(x+k_y\theta/2)}\delta(x'-x-k_y\theta) = e^{ik_x\frac{x+x'}{2}}\delta(x'-x-k_y\theta). \tag{4}$$

This is a bilocal expression, and we already see that its span along the x axis, $(x'-x)$, is proportional to the momentum along the conjugate y direction, i.e. $(x'-x) = \theta k_y$. In general, for n pairs of NC directions, one can keep only one coordinate out of every pair; commutativity is gained on the reduced space, at the expenses of strict locality. Using Eqs. (3) and (4) one sees that

$$< x'|\Phi|x >= \int \frac{dk_x}{2\pi\sqrt{2\omega_{k_x,k_y}}} \left[\hat{a}_{k_x,k_y} e^{i(\omega_{\vec{k}}t - k_x\frac{x+x'}{2})} + \hat{a}^{\dagger}_{k_x,-k_y} e^{-i(\omega_{\vec{k}}t + k_x\frac{x+x'}{2})}\right] \tag{5}$$

where $k_y = (x'-x)/\theta$. Thus, Φ annihilates a linear combination of rods of (arbitrary) momentum k_x and (fixed) length θk_y, and creates rods of momentum k_x and length $-\theta k_y$. It is not anymore a local operator, in contrast to usual field theory. Due to (1), one degree of freedom apparently disappears from (5). However it shows up through the modified dispersion relation

$$\omega_{(k_x, k_y = \frac{x'-x}{\theta})} = \sqrt{k_x^2 + \frac{(x'-x)^2}{\theta^2} + m^2}. \tag{6}$$

One notices the intrinsic IR/UV-dual character of the dipoles: both big momentum (UV) and big extension (IR) increase the energy. This second term reminds a string stretched between two separated D-branes.

Correlators

Two-point correlation functions for such dipoles are the VEV of the product of two bilocal fields (taken on the vacuum, $|0\rangle$, of the Fock space $\mathcal{F}$):

$$\langle 0| < x_4|\Phi|x_3 >< x_2|\Phi|x_1 > |0\rangle = \int \frac{dk_x}{8\pi^2\omega_{\vec{k}}} e^{ik_x[\frac{x_3+x_4}{2} - \frac{x_1+x_2}{2}]}\delta(x_4 - x_3 - x_2 + x_1). \tag{7}$$

Again, $k_y = (x'-x)/\theta$, $\omega_{\vec{k}} = \omega_{k_x,k_y}$ obeys (6), and there is no integral along k_y. If one compares (7) to the $(1+1)$-dimensional correlator of two commutative fields, $\langle 0|\phi(X_2)\phi(X_1)|0\rangle$, with $X_1 = (x_1+x_2)/2$ and $X_2 = (x_3+x_4)/2$, the differences are the $\frac{(x'-x)^2}{\theta^2}$ term in (6), and the delta function $\delta([x_4-x_3]-[x_2-x_1])$, which ensures that the length of the rod is conserved. Thus, our bilocal objects propagate in a $(1+1)-$dimensional space. The extra y direction is accounted for by their lenght, which contributes to the energy, and their orientation. Although we also call these rods dipoles, they do not necessarily have charges at their ends and they have extension in the absence of any background. Those rods may remind one about stretched open strings, or the double index representation of Yang-Mills theories.

Interactions

The quartic interaction term in (2) can be written as

$$\int dt Tr_{\mathcal{H}} V(\Phi) = \frac{g}{4!}\int dt \int_{x,a,b,c} < x|\Phi|a >< a|\Phi|b >< b|\Phi|c >< c|\Phi|x > . \tag{8}$$

To find the Feynman rules, we need the vacuum correlator (7), and a slight modification of the Dyson procedure. The basic 'vertex' for four-dipole scattering follows from

$$\langle -\vec{k}_3, -\vec{k}_4| : \int dt \int_{x,a,b,c} < x|\Phi|a >< a|\Phi|b >< b|\Phi|c >< c|\Phi|x >: |\vec{k}_1, \vec{k}_2\rangle. \tag{9}$$

$|\vec{k}_1, \vec{k}_2\rangle$ is a Fock space state with two quanta of momentum $\vec{k}_1$ and $\vec{k}_2$. The momenta $\vec{k}_{i,i=1,2,3,4}$ have each two components: $\vec{k}_i = (k_i, l_i)$. k_i is the momentum along x, whereas l_i represents the dipole extension along x (corresponding to the momentum along y) . Using Eq. (5) and integrating over x, a, b and c, one obtains the conservation laws $k_1 + k_2 + k_3 + k_4 = 0$ and $l_1 + l_2 + l_3 + l_4 = 0$. The final result differs from the four-point scattering vertex of $(2+1)$ commutative particles with momenta $\vec{k}_i = (k_i, l_i)$ only through the phase

$$e^{-\frac{i\theta}{2}\sum_{i<j}(k_i l_j - l_i k_j)}. \tag{10}$$

This is precisely the star-product modification of the usual Feynman rules. The phase (10) appears due to the bilocal nature of generic $< x'|\Phi|x >$'s.

By contracting various terms in (9), one obtains the one-loop corrections to the free rod propagator, together with the recipe for calculating loops. Again, the derivation is straightforward. The main point is that, in the end, one has to integrate over both the momentum and length of the dipole circulating in a loop. This $\frac{1}{2\pi}\int dk_{loop}\int dl_{loop}$ integration, together with the dispersion relation (6), brings back into play - especially as far as divergences are concerned - the y direction. It is easy to extend the above reasoning to $(2n+1)-$dimensions: unconstrained dipoles will propagate in a $(n+1)$-dimensional commutative space-time, with Feynman rules obtained as outlined above. Once the dipole lengths are interpreted as momenta in the conjugate directions, the rules are identical to those obtained long ago via star-product calculus.

IR/UV

We have derived directly from field theory the dipolar character of NC excitations; the momentum in the conjugate direction became the lenght of the dipole. A connection between UV and IR physics appeared naturally, and on a somehow more rigorous basis than in [6], for instance.

One can also view geometrically the differences between planar and nonplanar loop diagrams, and the role of low momenta in nonplanar graphs. To illustrate this, consider $(4+1)$-dimensions, $t, \hat{x}, \hat{y}, \hat{z}, \hat{u}$, with $[\hat{x}, \hat{y}] = [\hat{z}, \hat{w}] = i\theta$. In the $\{|x, z >\}$ basis, one has a commutative space spanned by the axes x and z, on which dipoles with momentum $\vec{p} = (p_x, p_z)$ and length $\vec{l} = (l_x, l_z) = \theta(p_y, p_w)$ evolve. During the scattering, four such dipoles meet in a four-edged poligon of area $\mathcal{A}$ (figure 1a).

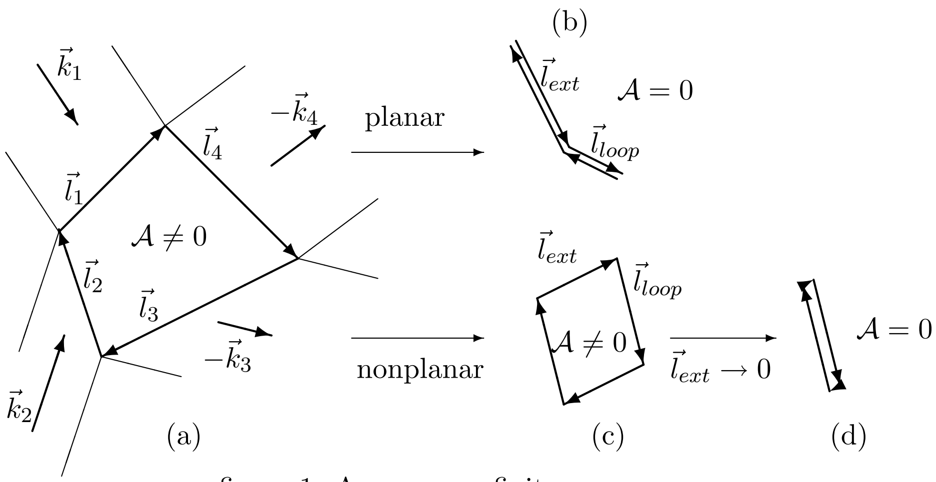

figure 1: Area versus finiteness

The one-loop correction to the propagator involves both planar and nonplanar diagrams. In the planar case, adjacent dipole fields are contracted. Momentum and length conservation enforce then the poligon to degenerate into a one-dimensional, zero-area object (figure 1b). UV divergences persist. In the nonplanar case, due to the nonadjacent contraction the area $\mathcal{A}$ does not go to zero (cf. figure 1c) unless the external dipole length vanishes (figure 1d). $\mathcal{A} \neq 0$ appears thus to be related to the disappearance of UV divergences. Actually, the true regulator is the phase (10). This is zero, i.e. ineffective, when $\mathcal{A} = 0$ in *both* the $|x, z >$ and $|y, u >$ bases. That corresponds to zero external length *and* momentum in the dipole picture, which means that the resulting divergence is half IR ($\vec{p}_{ext} = 0$)

and half UV ($\vec{l}_{ext} = 0$)! In Weyl space this is just the usual zero external momentum, say $p_\mu^{ext} = 0$, and one speaks about an IR divergence. For dipoles the divergence comes from having zero vertex area $\mathcal{A}$ in any basis, and is half IR and half UV. NCFT appears to be somehow in between usual FT and string theory: when the interaction vertex is a point, UV infinities appear; when it opens up, as in string theory, amplitudes are finite.

Causality

Causality is an essential feature of any physical model. In usual field theory, it is defined via the ordering of precise events taking place in space-time. For a field ϕ depending on space-time coordinates $x^\mu = (x^0 = t, x^1, x^2, x^3)$, the (micro)causality condition reads

$$[\phi(x), \phi(0)] = 0, \qquad t^2 - \vec{x}^2 \leq 0. \tag{11}$$

It reflects the assumption that two events having space-like separation cannot influence each other.

In the NCFT literature causality is rarely discussed, and even then with contradictory results. The reason is the use of the Weyl-Moyal quantization procedure, in which NC space is mapped to the commutative space of Weyl symbols, whose correspondence to physical space (assumed NC) is at best statistical. Thus, a "point" in Weyl symbol space has no precise correspondent in the physical (NC) space. On the other hand the product of functions gets deformed to the Moyal star product

$$f(x) \cdot g(x) \quad \rightarrow \quad f(x) \star g(x) \equiv \lim_{y \to x} \exp\left(\frac{i}{2}\theta^{\mu\nu}\partial_\mu^x \partial_\nu^y\right) f(x)g(y). \tag{12}$$

In consequence, if one wants to generalize the causality condition (11) to NC fields, one encounters two ambiguities.

1. It is not clear whether one should take the commutator or the star-commutator of two fields [7].

2. It is not clear what a "space-like interval" means when some of the coordinates are noncommuting - hence not all can be sharply measured simultaneously. Several conditions were used in the literature, for events separated by the quadri-vector $(\Delta t, \Delta\vec{r})$ *in Weyl space*:

a) the usual light-cone condition: $\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \leq 0$, too weak to always ensure vanishing of the commutator of two fields.

b) the light-wedge condition: $\Delta t^2 - \Delta z^2 \leq 0$ [8, 9], which drops completely the noncommuting coordinates; it is too strong, since it operates only on the commuting part of the space; it cannot be implemented if no commuting coordinate z is available.

c) an intermediate condition: $\Delta t^2 - \Delta z^2 \leq 2\theta$ [10], with the RHS trying to account for the average spreading $\Delta x^2 + \Delta y^2$. It involves a statistical correction with respect to the condition (b) above and is still inappropriate [10], as we will also see.

We will remove both ambiguities, using the canonical framework of [4, 5]. The first ambiguity is disposed of simply by using an operatorial formulation, in which the commutator is uniquely defined. The second, more vexing, problem is solved since, as it will be shown, it is natural to drop *one* of the noncommuting coordinates, say y, and then require zero commutator provided $\Delta t^2 - \Delta x^2 - \Delta z^2 \leq 0$, but in *physical space*, not in Weyl space. In consequence, we will show - disproving previous claims, that NC theories with commuting time *are* causal.

It has been shown in [4] that *free* NC fields behave exactly like commutative fields living in a lower-dimensional space. In fact a free $(1+1)$-dimensional dipole [resulting from the $2+1$ NC theory we were discussing] with endpoints situated at x and x' behaves like a commutative $(1+1)$ point particle centered at $\frac{x+x'}{2}$, but with a modified dispersion relation $\omega^2 = k_x^2 + \frac{(x-x')^2)}{\theta^2}$. In conclusion all usual manipulations performed on propagators in $(1+1)$-dimensions can be carried over, including those used to demonstrate causality. This immediately shows that at the free level NC field theories are causal, contrary to previous claims [7].

For interacting fields, one expects the dipolar character of the degrees of freedom to manifest, as e.g. in perturbation theory [4]. It is however remarkable that as far as causality issues are concerned, bilocality has little influence, and a proof of causality can be given like for commutative theories. For, consider the vanishing of the commutator to hold,

$$[\phi(t_1, \bar{x}_1), \phi(t_2, \bar{x}_2)] = 0 \tag{13}$$

with $\bar{x}_1 = \frac{x_1+x_1'}{2}$, $\bar{x}_2 = \frac{x_2+x_2'}{2}$ being the average positions (centers of mass) of the two dipoles considered. We want (13) to be true when the interval, defined with respect to the average dipole positions, is space-like,

$$(t_1 - t_2)^2 - (\bar{x}_1 - \bar{x}_2)^2 \leq 0. \tag{14}$$

Equations (13, 14) are however generically equivalent to

$$[\phi(t, \bar{x}), \phi(t, \bar{y})] = 0, \qquad \vec{x} \neq \vec{y}, \tag{15}$$

provided one can apply a boost along x to render equal the two times appearing in Eq. (13).

This requires the $(1+1)$-dimensional dipole theory to be invariant under boosts in the x-direction (a fact completely overlooked in the literature, which claims that the only invariance left after NC is imposed is a product of $O(2)$ for the NC part and of the Lorentz group, e.g. $SO(1,1)$, for the rest). The invariance follows from the form of the classical Lagrangian for dipoles,

$$2L = (\partial_t \phi)^2 - (\partial_{\bar{x}} \phi)^2 - [(\theta^{-1} \Delta x)^2 + m^2] \phi^2 - 2V(\phi). \tag{16}$$

Above, $V(\phi)$ is the potential for the field, $\bar{x}$ generically denotes the average position of a dipole, $\frac{x+x'}{2}$, whereas Δx denotes its span, $x - x'$. The only thing to prove is the invariance of the third term in the RHS. which immediately follows from the tensorial character of the inverse of $\theta = \theta_{xy} \sim xy$ and the usual Lorentz transformation of Δx. At the quantum level one has no reason to worry about an anomaly, since the integration measure in the path integral is invariant. In consequence, Eqs. (13, 14) are tantamount, via a boost, to

$$e^{iH't} [\phi(0, \bar{x}), \phi(0, \bar{y})] e^{-iH't} = 0 \tag{17}$$

which is true at $t = 0$, since this is by definition the time at which the fields behave like free fields (H' denotes the interacting part of the Hamiltonian, including V, in the interaction representation). We stress that the above causality argument works for a fully interacting theory.

Adding now the (passive) commutative coordinate z, we conclude that the correct criterion for causality is

$$[\phi(t_1, \bar{x}_1, z_1), \phi(t_2, \bar{x}_2, z_2)] = 0, \qquad (t_1 - t_2)^2 - (\bar{x}_1 - \bar{x}_2)^2 - (z_1 - z_2)^2 \leq 0, \tag{18}$$

and that it *is* satisfied in NC field theory.

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References

[1] M.R. Douglas and N.A. Nekrasov, Rev.Mod.Phys. **73** (2002) 977; R.J. Szabo, hep-th/0109162; J.A. Harvey, hep-th/0102076. These reviews include comprehensive lists of references.

[2] A review advocating the dipole picture is given by Soo-Jong Rey, hep-th/0207108. Additional original references can be found in [1] and [4].

[3] N. Seiberg, L. Susskind and N.Toumbas, JHEP **06**, 44 (2000).

[4] C.S. Acatrinei, Phys.Rev. **D67**, 045020 (2003).

[5] C.S. Acatrinei, J. Mod. Phys. **A41**, 215401 (2008).

[6] A. Matusis, L. Susskind and N. Toumbas, JHEP **0012** (2000) 002.

[7] O.W. Greenberg, Phys. Rev. **D73**, 045014 (2006).

[8] L. Alvarez-Gaume and M. Vazquez-Moto, Nucl. Phys. **B668**, 293 (2003).

[9] C.S. Chu, K. Furuta and T. Inami, Int.J.Mod.Phys. **A21**, 67 (2006).

[10] A. Haque and S. D. Joglekar, J. Mod. Phys. **A41**, 215402 (2008).