Physics AUC, vol. 18, 120-129 (2008) **PHYSICS AUC**

Numerical simulations with Ricci flow on surfaces A review and some recent results*

Dumitru N. Vulcanov
West University of Timişoara
Department of Theoretical and Applied Physics
B-dul V. Parvan no. 4, Timişoara, Romania

Abstract

The article presents the main recent results concerning numerical simulations using geometric flow equations, as the Ricci flow (RF) and other related flows. Although RF equations are simpler than Einstein equations their numerical integration open new and striking problems which are pointed out in this article.

1 Introduction - The Ricci flow

The Ricci flow [1],[4],[5] is a process deforming the metric of a Riemannian manifold in a manner formally analogous to the diffusion of heat, smoothing out irregularities in the metric. It plays an important role in the apparent proof of the Poincare conjecture, one of the seven Millennium Prize Problems for which the Clay Mathematics Institute offers a 1,000,000 USD prize for a correct solution.

Given a Riemannian manifold with metric tensor g_{ij}, we can compute the Ricci tensor R_{ij}, which collects averages of sectional curvatures into a kind of "trace" of the Riemann curvature tensor. If we consider the metric tensor (and the associated Ricci tensor) to be functions of a variable which is usually called "time" (but which may have nothing to do with any physical time), then the Ricci flow may be defined by the geometric evolution equation

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} \tag{1}$$

The normalized Ricci flow is the equation

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} + \frac{2}{n} r g_{ij} \tag{2}$$

where r is the average (mean) of the scalar curvature (which is obtained from the Ricci tensor by taking it's trace) and n is the dimension of the manifold. This normalized equation preserves the volume of the metric. Ricci flow expands negatively curved regions of the manifold, and contract positively curved regions.

Here are some examples of spaces under Ricci flow [5]:

- Euclidean space, or more generally (Ricci) flat; Ricci flow has no effect.
- Sphere ; Ricci flow collapses the manifold to a point in finite time.
- Einstein manifold (Ricci = constant×metric) ; Ricci flow will collapse it to a point if it has positive curvature, leave it invariant if it has zero curvature, and expand it if it has negative curvature.

*Based on the talk "Numerical investigations in Ricci flow on 2 and 3 dimensional surfaces" given at the 6-th Spring School and Workshop on QFT and Hamiltonian Systems, Calimanesti-Caciulata, Romania, May 6-11, 2008

- In particular, this shows that in general the Ricci flow cannot be continued for all time, but will produce singularities. For a 3 dimensional manifold, Perelman showed how to continue past the singularities using surgery on the manifold.

- Another more non-trivial example is the Cigar soliton (or Witten black-hole) having the metric as

$$g = \frac{dx^2 + dy^2}{1 + x^2 + y^2} = \frac{dr^2 + r^2 d\theta^2}{1 + r^2} = ds^2 + tanh^2(s) d\theta^2$$

  where $s = arcsinh(r) = log\left(1 + \sqrt{1 + r^2}\right)$. This is a stationary solution to the Ricci flow (steady soliton), has positive curvature, is asymptotic to a cylinder, its curvature decays exponentially and is rotationally symmetric.

Many variants of the Ricci flow have also been studied:

- Various curvature flows defined using either an extrinsic curvature, which describes how a curve or surface is embedded in a higher dimensional flat space, or an intrinsic curvature, which describes the internal geometry of some Riemannian manifold,

- Various flows which extremalize some quantity mathematically analogous to an energy or entropy,

- Various flows controlled by a p.d.e. which is a higher order analog of a nonlinear diffusion equation.

2 Numerical simulations - Why ?

The problem of numerical simulations and visualizations with Ricci flow of 2 or 3 dimensional manifolds comes naturally as Ricci flow is geometric by nature.

Ricci flow acts directly on the metric of the surface, tending not to preserve the embeddedness. A number of interesting results have been obtained by restricting to classes of metrics of revolution, since such symmetries are preserved under Ricci flow and the metric depends on considerably fewer parameters in such cases. These surfaces tend to remain embedded in $\mathbf{R}^3$ making direct visualization possible.

Since the Ricci flow equations (1) or (2) are only **weekly parabolic** and since numerical evolutions appear to be very unstable, we need some solutions to avoid this. Mainly there are in the literature two solutions (see bellow), namely :

- filtering and reparamatrization method (inspired by spectral methods)
- using of other flow equations derived from (1), the deTurck flow is such an example.

However an explicit finite-difference scheme for (1) is still usable with very small time-steps and high number digits precision (several hundred) - removing short wavelength instabilities.

The main goal of numerical simulations with Ricci flow is to exhibit the formation of singularities and/or neck pinching phenomenon, which occurs naturally for metrics of revolution. Good estimates are available to understand the limiting behavior of metrics as they become singular under neck pinching [2], [3].

Numerical simulations in Ricci flow are very similar to those in numerical relativity (the main equations here are the Einstein equations in the 3+1 split [9] of spacetime being similar to Ricci flow equations), thus some methods and experience can be ... imported here - even codes.

There are not very many results in this direction reported till now. We can mention only two articles, namely :

1) Garfinkle & Isenberg - math.DG/0306129 (GI) [2]
2) Rubinstein & Sinclair - math.DH/0406189 (RS) [3]

We will concentrate now on these reported results in the next of the article, pointing out the numerical methods and results and showing of some of the results. Then we will briefly summarize some of our own results which will be reported and published elsewhere.

A special mention for those who are searching around the literature of the topic : there are computer graphics researches which involves Ricci flow [8]. Computer graphics applications with Ricci flow were developed (mainly at SUNY and Harvard) by S.T. Yau, F. Luo, D.X. Gu and others. Ricci flow provides a mathematical tool to compute the desired metric that satisfies prescribed curvatures. A Ricci flow algorithm was developed (B. Chow and F. Luo) based on the so called "combinatorial Ricci flow theory" The convergence and stability of this Ricci flow algorithm was proved. Tackling shape representation and deformation in computer graphics with Ricci flow is thus done, with application in different areas, some of them involving security and intelligence (face recognition applications for example).

3 Corseted sphere geometries

The corseted sphere geometries ([2]) and their flows are all represented by spherically symmetric metrics on S^3 of the form

$$g = e^{2X}\left(e^{-2W}d\psi^2 + e^{2W}\sin^2\psi[d\theta^2 + \sin^2\theta d\phi^2]\right) \tag{3}$$

Here (ψ, θ, ϕ) are standard angular coordinates on the three sphere. The metric functions X and W are functions only of ψ. $W = X$ and X is choose so that

- $4e^{4X}\sin^2\psi = \sin^2 2\psi$ for $\cos^2\psi \geq 1/2$
- $4e^{4X}\sin^2\psi = \sin^2 2\psi + 4\lambda\cos^2 2\psi$ for $\cos^2\psi \leq 1/2$.

Here λ is a constant, which parameterizes the degree of corseting for these geometries.

For their numerical investigations, Garfinkle and Isenberg (GI) [2] used the DeTurck flow, where the PDE equations are strong parabolic :

$$\partial_t \hat{g}_{ab} = -2\hat{R}_{ab} + 2\hat{D}_{(a}V_{b)} \tag{4}$$

where $\hat{D}_a$ is the derivative operator associated with the metric $\hat{g}_{ab}$, and where the vector field V^a is given by

$$V^a = \hat{g}^{bc}\left(\hat{\Gamma}^a_{bc} - \Delta^a_{bc}\right) \tag{5}$$

with $\hat{\Gamma}^a_{bc}$ being the connection of the metric $\hat{g}_{ab}$ and with Δ^a_{bc} being any fixed connection.

GI applied a direct finite differences schema for the time evolution and centered finite differences for the spatial derivatives in (4). The equations were written in terms of the variable $S = W/sin^2(\psi)$ to avoid the numerically imposition of two conditions on W at a single point (at the poles W must vanish).

As the evolution equations have the form $\partial_t F(\psi, t) = G(\psi, t)$ they can numerically evolved using the approximation

$$F_i^{n+1} = F_i^n + \Delta t G_i^n \tag{6}$$

at the time-step $n+1$. Here $1 \leq i \leq N$, N being the number of points on the spatial grid.

The numerical results pointed out a critical value for the λ parameter ($\lambda = 0.1639$) dividing the behavior under the flow into two regimes, namely :

- a sub-critical one - for larger λ the flow converges to a round sphere metric
- a super-critical one - for smaller λ the flow goes to a S^2 neck pinching singularity

Some of the numerical results obtained by GI are showed in the figures no. (1) - (3) below, reproduced with D. Garfinkle kind permission from [2].

At the critical value for λ the flow approaches a "javelin" geometry, marked by curvature singularities at the poles, with roughly uniform curvature between the poles. This javelin geometry corresponds to the "type 3" singularity described by Hamilton.

The work of Garfinkle and Isenberg is a typical example of how the experience in numerical relativity can be applied to Ricci flow simulations. The DeTurk trick used to make the PDE strongly parabolic is similar to the ADM-BSSN version for numerical relativity [9].

In spite of this, we still think that the initial Ricci flow equations (normalized or not) can be used, as the explicit finite differences schema can be done stable - a Courant factor in certain conditions is possible to exhibit for this schema, as we proved using special Maple programs (see below). This is similar to the pure ADM method in numerical relativity. We will develop this idea in our next investigations. But this is an open question from now one, and the results will be reported elsewhere.

A special mention on the results reported by GI. They not offer any idea about the stability and convergence tests they done (if so) so even in the case we want to reproduce and continue in this line, we think is necessary first to have a stability analysis of the numerical schema used (the above one or any other one) and a careful check of the convergence.

4 2-dimensional surfaces of revolution

In their work on numerical simulations with Ricci flow, Rubinstein and Sinclair (RS) [3] were more concentrated on the visualization problem of 2 or 3 dimensional surfaces of revolution. Thus they first pointed out two important theorems :

Theorem 1 *Suppose that a metric of the form $ds^2 + \alpha^2(s)d\theta^2$ is chosen on S^n. Then there is an isometric embedding as a manifold of revolution in $\mathbf{R}^{(\mathbf{n+1})}$ if and only if the same metric, viewed as on S^2, can be isometrically embedded into $\mathbf{R}^3$ as a surface of revolution.*

Theorem 2 *If there is an isometric embedding of S^n as a manifold of revolution in $\mathbf{R}^{(n+1)}$ then the manifold remains isometrically embedded so long as Ricci flow gives a smooth solution.*

A two-dimensional surface of revolution of genus zero embedded in $\mathbf{R^3}$ can be defined in a polar representation with coordinates $x^1 = \rho$ and $x^2 = \theta \in [0, 2\pi[$ by a metric of the form

$$[g_{\mu\nu}] = \begin{bmatrix} h(\rho) & 0 \\ 0 & m(\rho) \end{bmatrix},$$

where $\sqrt{m(\rho)}$ has the direct physical interpretation as the radius from the axis of rotation. For a closed surface, we have

$$m(\rho_{pole}) = 0.$$

choosing $\rho_{\text{North\_Pole}} = 0$ and $\rho_{\text{South\_Pole}} = \pi$.

To avoid the numerical instabilities in an implicit scheme for (1) and inspired by spectral methods, RS introduced a filter which consists of transforming to Fourier space (DFT), dropping shorter wavelength terms, and then transforming back.

$$h(\rho) = \sum_{i=0}^{N_h} h_i \cos 2i\rho \quad \textit{and} \quad \sqrt{m(\rho)} = \sum_{i=0}^{N_m} m_i \sin\left((2i+1)\rho\right),$$

where equality of course only applies to $N_h = N_m = \infty$ [3].

The Ricci flow forces some parts of a surface to contract while others are inflating creating further sources of numerical instability. The solution to this problem is to re-parametrize the metric, the most pleasant one let $h(\rho)$ be a constant. This corresponds to ρ being directly proportional to distance from a pole :$\ell(\rho) = \int_{s=0}^{\rho} \sqrt{h(s)}\, ds$. Then we have [3]

$$[g_{\mu\nu}] = \begin{bmatrix} h(\rho) & 0 \\ 0 & m(\rho) \end{bmatrix} \mapsto \begin{bmatrix} \left(\frac{\ell(\pi)}{\pi}\right)^2 & 0 \\ 0 & m\left(\ell^{-1}\left(\frac{\rho \times \ell(\pi)}{\pi}\right)\right) \end{bmatrix}$$

where $\rho \in [0, \pi]$ both before and after.

In their numerical investigations RS used a 2-dimensional dumbbell shape surface where at the initial time having :

$$h(\rho) = 1 \quad ; \quad m(\rho) = \left(\frac{sin(\rho) + c_3 sin(3\rho) + c_5 sin(5\rho)}{1 + 3c_3 + 5c_5} \right)^2$$

with c_3 and c_5 two constants.

A special code (Ricci_rot) - written in C, was used for numerical simulations and an OpenGL standard software to visualize the results. The behavior in all cases cases were what one expects – flow towards spheres of constant positive Gaussian curvature. Some of the results RS reported are visualized in figure no. (4) reproduced here from [3] (as the next ones (5) and (6)) with the kind permission of H. Rubinstein.

5 3-dimensional surfaces of revolution

For the 3-dimensional surface of revolution RS choose the metric as

$$[g_{\mu\nu}] = \begin{bmatrix} h(\rho) & 0 & 0 \\ 0 & m(\rho) & 0 \\ 0 & 0 & m(\rho)\cos^2\left(\sqrt{K_2}\,\theta\right) \end{bmatrix}, \tag{7}$$

where $\rho \equiv x^1$ plays the role of a latitude and $\theta \equiv x^2$ the role of a latitude on the abstract Riemannian surface of revolution. As initial date RS used a 3-dimensional surface of revolution with :

$$m(\rho) = \frac{1}{10000} + \sin^2\left(\frac{9\,\pi\,\rho}{40}\right) \quad h(\rho) = 1 \quad \text{and} \quad K_2 = 1$$

The pinching behavior of this surface under unnormalized Ricci flow (1) was studied starting at $t = 0$. The time at which pinching occurs ($m(0) = 0$) was denoted by $t = T$.

This time an explicit finite-differences schema was used. The instabilities forced to restrict to fairly large time steps. Some of the instabilities were solved doing the simulations in Maple with several hundred digits precision. This open the line for future simulations in Maple using alternative Ricci flow equations.

The method was not verified for convergence and thus these data are called by the authors as "qualitatively correct". However the results are compatible with the known analytical properties. A "pinching" time was identified (t=000077343750). Speculations that the flow pinching creates two independent geometric bodies are suggested and ideas how to treat this situation for later time. This is obvious from the next figure no. (5) from [3], where the lines are taken at **unequal** time intervals. The axis of rotation is horizontal.

A second approach to studying pinching involves series expansions of the metric components h and m. Expanded them to tenth order in ρ (even terms only) and first order in t :

$$h(\rho, t) = \sum_{i=0}^{10} (h_i + \dot{h}_i\, t)\, \rho^i \quad \text{and} \quad m(\rho, t) = \sum_{i=0}^{10} (m_i + \dot{m}_i\, t)\, \rho^i$$

Equations for the quantities $\dot{h}_i$ and $\dot{m}_i$ are thus obtained through the flow equations. These were processed with Maple and Maple generated C codes using Euler's method with a very primitive form of adaptive step-size estimation. A curve-fitting to the previous obtained data procedure was developed on this basis. This is illustrated in the next figure no. (6) - [3].

6 Some recent results in using Maple for Ricci flow simulations

Having in mind the main results and problems we pointed out in the previous sections we investigated the use of Maple platform in doing numerical simulations and visualizations in Ricci flow. Why using Maple for numerical simulations in Ricci flow ?

Because Maple is an integrated platform which can do symbolic computations, numerical computing and visualizations in the same .. worksheet ! Thus we done all the necessary steps for numerical simulations in Ricci flow using only Maple, namely :

- (1) calculations of the differential equations in several cases we investigated
- (2) finite differencing of the PDE obtained in the previous step
- (3) stability and convergence analysis on the finite differences schema used in order to establish a Courant factor for stable numerical evolution
- (4) analysis of initial data
- (5) the numerical evolution of the finite differences equations in question
- (6) visualization of results - even movies !!

  The main advantages are :

- Storing the analytical and numerical results in separate Maple libraries, for later use. Example : step (1) above produce the main equations stored in libraries loaded later in separate Maple worksheets.
- as a result steps (2) and (3) or (4) above can be done separately in special worksheets separately doing the numerical evolution and visualization after analysis steps.
- Everything was done under the same language and environment !
- Easy to use results for anybody familiar with Maple and not only !

A special mention for the step (1) above : here we used the GrtensorII package [7] for algebraic computing of the main equations and geometrical objects (as the Ricci tensor and scalar). GrTensorII is specially designed for differential geometry calculations in Riemannian geometry

Here we shall point some of the numerical results we obtained in Maple for a two-folded surface of revolution and a grid with 50 points having a size of the spatial step as $\Delta\rho = \delta = \pi/n = 0.06283185308$ and a time-step taken as $\Delta t = dete = 0.002$.

In the left plot from the figure no. (7) we have the time evolution of the radius of the surface (distance from the rotation axis) and the right plot is the Ricci scalar, both after 96 iterations.

Next plots from the figure no. (8) represents the 3D shape of our two-folded surface at the initial time and after 96 iterations.

This proves at least the feasibility of the numerical simulations and visualizations using Maple for deforming surfaces of revolution under Ricci flow. As we done a complete stability and convergence analysis of different numerical schemes we used in our Maple programs, a complete description and all details will be included in a future article to be published.

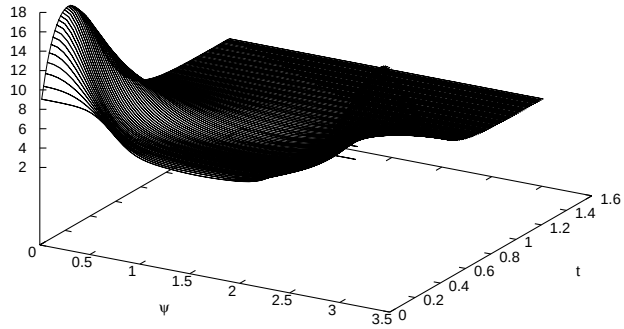

Figure 1: R_{S^2} for sub-critical Ricci flow - reproduced from [2] - courtesy of D. Garfinkle

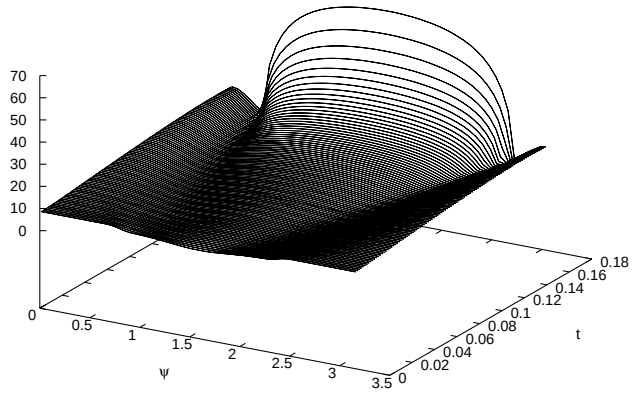

Figure 2: R_{S^2} for super-critical Ricci flow - reproduced from [2] - courtesy of D. Garfinkle

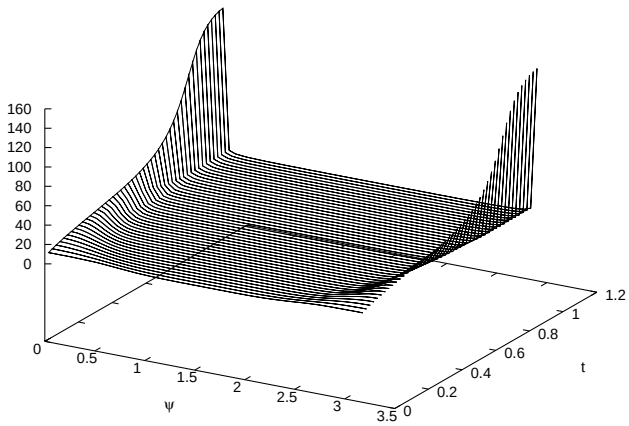

Figure 3: R_{S^2} for critical Ricci flow - reproduced from [2] - courtesy of D. Garfinkle

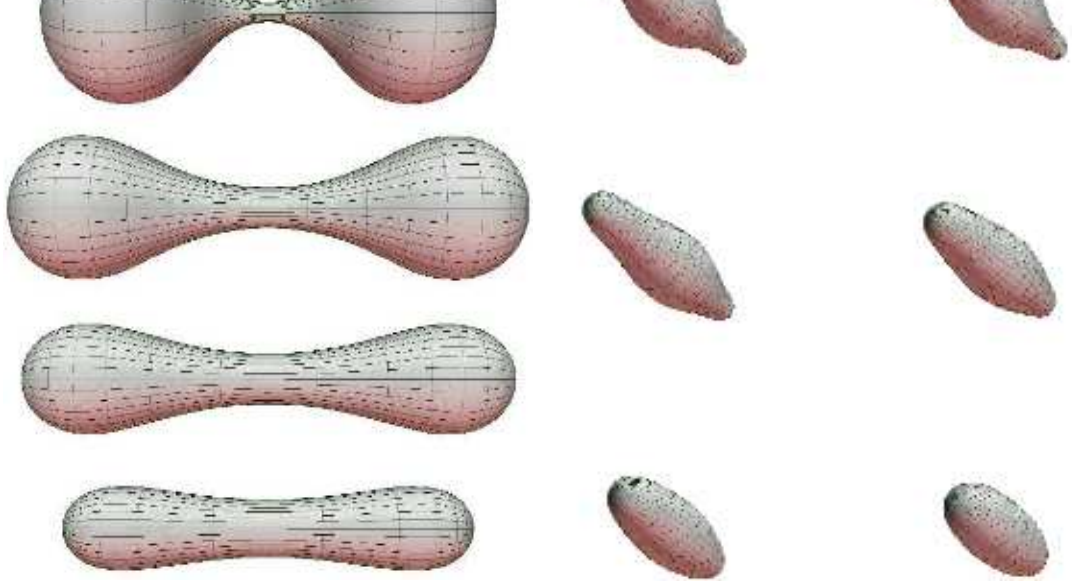

Figure 4: Dumbbell shape surface of revolution under Ricci flow - reproduced from [3] courtesy of H. Rubinstein

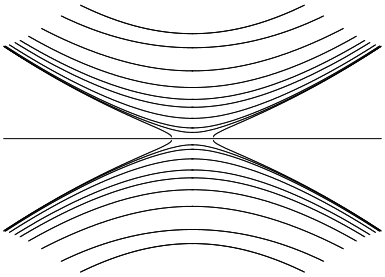

Figure 5: Qualitatively correct deformation of the cross-section of a 3-dimensional surface of revolution - reproduced from [3] courtesy of H. Rubinstein

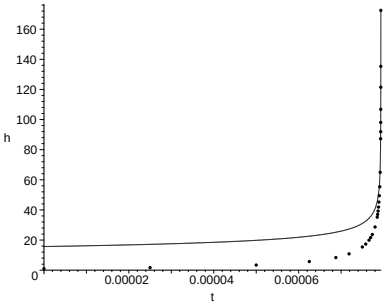

Figure 6: Evolution of h. The points are numerical data - reproduced from [3] courtesy of H. Rubinstein

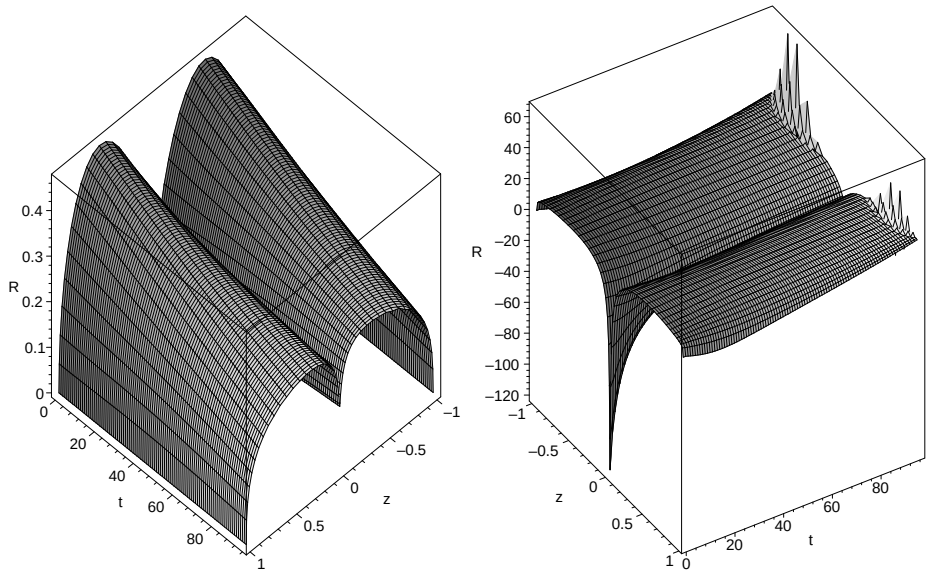

Figure 7: Time evolution of the radius of rotation $\sqrt{m(t)}$ (left) and of the Ricci scalar after 96 iterations

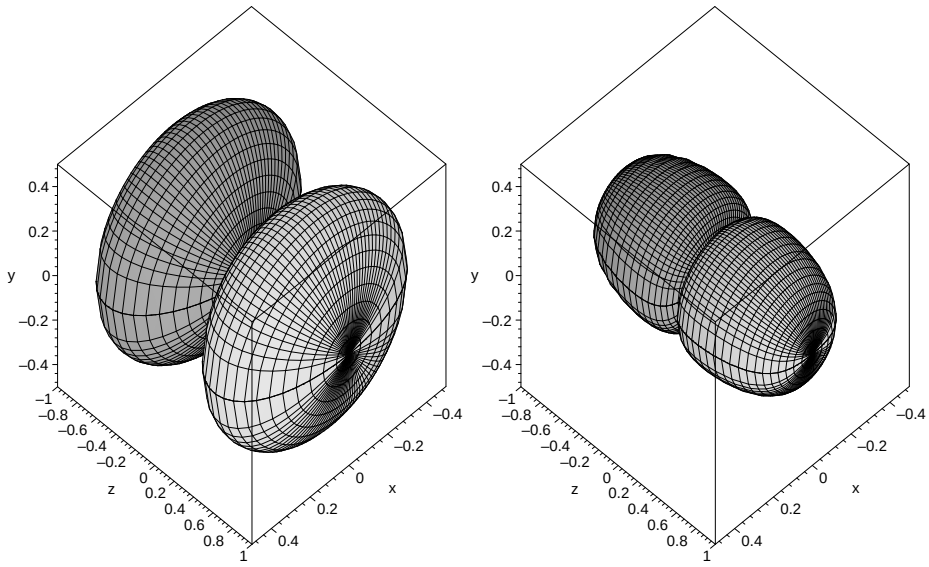

Figure 8: The 3D shape of the two folded surface at the initial time (left) and after 96 iterations (right)

7 Acknowledgments

Special thanks to David Garfinkle and Hyam Rubinstein for their kind permission to reproduce the pictures I included here

This work was supported under Grant CEEX nr. 2-CEx06-11-92/19.09.2006 with the Romanian Ministry of Education and Research

References

[1] R.S. Hamilton, J. Diff. Geom., 17, 255 (1982)

[2] D. Garfinkle, J. Isenberg, math.DG/0306129 (2003)

[3] J.H. Rubinstein, R. Sinclair, Exp.Math., 14, nr.3 (2005), math.DG/0506189

[4] B. Chow, *A Survey of Hamilton's Program for the Ricci flow on 3-Manifolds*, math.DG/0211266 (2002)

[5] P. Topping, Lectures on Ricci flow, Cambridge Univ. Press, Cambridge (2006)

[6] D. DeTurck, J. Diff. Geom., 18, 157 (1983)

[7] **http://grtensor.org**

[8] **http://www.cvc.sunnysb.edu** and **http://www.cise.ufl.edu/ gu** David X. Gu personal web page

[9] M. Alcubierre, Introduction to (3+1) numerical relativity, Clarendon Pres, Oxford (2007)