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Gauge Invariance in the Causal Approach - the Renormalization of Quantum Electrodynamics

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Abstract

Quantum theory of the gauge models in the causal approach leads to some cohomology problems. We investigate these problems in detail for quantum electrodynamics and give a rigorous proof for the renormalization of quantum electrodynamics.

1 Introduction

The general framework of perturbation theory consists in the construction of the chronological products such that Bogoliubov axioms are verified [1], [5], [4], [10]; for every set of Wick monomials $W_1(x_1), \dots, W_n(x_n)$ acting in some Fock space $\mathcal{H}$ one associates the operator $T^{W_1,\dots,W_n}(x_1, \dots, x_n)$; all these expressions are in fact distribution-valued operators called chronological products. It will be convenient to use another notation: $T(W_1(x_1), \dots, W_n(x_n))$. The construction of the chronological products can be done recursively according to Epstein-Glaser prescription [5], [6] (which reduces the induction procedure to a distribution splitting of some distributions with causal support) or according to Stora prescription [12] (which reduces the renormalization procedure to the process of extension of distributions). These products are not uniquely defined but there are some natural limitation on this arbitrariness. If the arbitrariness does not grow with n we have a renormalizable theory.

Gauge theories describe particles of higher spin. Usually such theories are not renormalizable. However, one can save renormalizablility using ghost fields. Such theories are defined in a Fock space $\mathcal{H}$ with indefinite metric, generated by physical and un-physical fields (called *ghost fields*). One selects the physical states assuming the existence of an operator Q called *gauge charge* which verifies $Q^2 = 0$ and such that the *physical Hilbert space* is by definition $\mathcal{H}_{\rm phys} \equiv Ker(Q)/Ran(Q)$. The space $\mathcal{H}$ is endowed with a grading (usually called *ghost number*) and by construction the gauge charge is raising the ghost number of a state. Moreover, the space of Wick monomials in $\mathcal{H}$ is also endowed with a grading which follows by assigning a ghost number to every one of the free fields generating $\mathcal{H}$. The graded commutator d_Q of the gauge charge with any operator A of fixed ghost number

$$d_Q A = [Q, A] \tag{1.1}$$

is raising the ghost number by a unit. It means that d_Q is a co-chain operator in the space of Wick polynomials. From now on $[\cdot, \cdot]$ denotes the graded commutator.

A gauge theory assumes also that there exists a Wick polynomial of null ghost number $T(x)$ called *the interaction Lagrangian* such that

$$[Q, T] = i\partial_\mu T^\mu \tag{1.2}$$

for some other Wick polynomials T^μ. This relation means that the expression T leaves invariant the physical states, at least in the adiabatic limit. Indeed, if this is true we have:

$$T(f)\ \mathcal{H}_{\rm phys}\ \subset\ \ \mathcal{H}_{\rm phys} \tag{1.3}$$

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up to terms which can be made as small as desired (making the test function f flatter and flatter). In the case of quantum electrodynamics we also have:

$$[Q, T^{\mu}] = 0 \tag{1.4}$$

but in more general models we have a divergence in the right hand side, etc.

We can also use a compact notation T^I where I can be $I = \emptyset$ or $I = \mu$ All these polynomials have the same canonical dimension

$$\omega(T^I) = 4, \ \forall I \tag{1.5}$$

and because the ghost number of $T \equiv T^{\emptyset}$ is null, then we also have:

$$gh(T^I) = |I|. \tag{1.6}$$

We can write (1.2) and (1.4) in the compact form

$$d_Q T^I = i \frac{\partial}{\partial x^{\mu}} T^{I\mu} \tag{1.7}$$

with the convention $T^I = 0, \ |I| \geq 2$. Now we can construct the chronological products

$$T^{I_1,\dots,I_n}(x_1,\dots,x_n) \equiv T(T^{I_1}(x_1),\dots,T^{I_n}(x_n))$$

according to the recursive procedure. We say that the theory is gauge invariant in all orders of the perturbation theory if the following set of identities generalizing (1.7):

$$d_Q T^{I_1,\dots,I_n} = i \sum_{l=1}^{n} (-1)^{s_l} \frac{\partial}{\partial x_l^{\mu}} T^{I_1,\dots,I_l\mu,\dots,I_n} \tag{1.8}$$

are true for all $n \in \mathbb{N}$ and all $I_1,\dots,I_n$. Here we have defined

$$s_l \equiv \sum_{j=1}^{l-1} |I|_j \tag{1.9}$$

(see also [3]). In particular, the case $I_1 = \ldots = I_n = \emptyset$ it is sufficient for the gauge invariance of the scattering matrix, at least in the adiabatic limit: we have the same argument as for relation (1.3).

Such identities can be usually broken by *anomalies* i.e. expressions of the type $A^{I_1,\dots,I_n}$ which are quasi-local and might appear in the right-hand side of the relation (1.8). These expressions verify some consistency conditions - the so-called Wess-Zumino equations. One can use these equations in the attempt to eliminate the anomalies by redefining the chronological products. All these operations can be proved to be of cohomological nature.

If one can choose the chronological products such that gauge invariance is true then there is still some freedom left for redefining them. To be able to decide if the theory is renormalizable one needs the general form of such arbitrariness. Again, one can reduce the study of the arbitrariness to descent equations of the type as (1.7).

Such type of cohomology problems have been extensively studied in the more popular approach to quantum gauge theory based on functional methods (following from some path integration method). In this setting the co-chain operator is non-linear and makes sense only for classical field theories. On the contrary, in the causal approach the co-chain operator is linear so the cohomology problem makes sense directly in the Hilbert space of the model. One needs however a classical field theory machinery to analyze the descent equations more easily.

In this paper we want to give a general description of these methods for quantum electrodynamics. In the next Section we give a rigorous definition for the quantum electrodynamics model with ghost fields. Then in Section 3 we remind the axioms verified by the chronological products and consider the particular case of gauge models. In Section 4 we determine the cohomology of the operator d_Q. In Section 5 we use this cohomology and we can prove gauge invariance of quantum electrodynamics for all orders of perturbation theory.

2 Quantum Electrodynamics

We consider a vector space $\mathcal{H}$ of Fock type generated (in the sense of Borchers theorem) by the vector field v_μ (with Bose statistics) and the scalar fields $u, \tilde{u}$ (with Fermi statistics). The Fermi fields are usually called *ghost fields*. We suppose that all these (quantum) fields are of null mass. Let Ω be the vacuum state in $\mathcal{H}$. In this vector space we can define a sesquilinear form $< \cdot, \cdot >$ in the following way: the (non-zero) 2-point functions are by definition:

$$< \Omega, v_\mu(x) v_\nu(y) \Omega >= i\ \eta_{\mu\nu}\ D_0^{(+)}(x-y)\ \cdot I,$$
$$< \Omega, u(x) \tilde{u}(y) \Omega >= -i\ D_0^{(+)}(x-y)\ \cdot I \qquad < \Omega, \tilde{u}(x) u(y) \Omega >= i\ D_0^{(+)}(x-y)\ \cdot I \tag{2.1}$$

and the n-point functions are generated according to Wick theorem. Here $\eta_{\mu\nu}$ is the Minkowski metrics (with diagonal $1, -1, -1, -1$) and $D_0^{(+)}$ is the positive frequency part of the Pauli-Villars distribution D_0 of null mass. To extend the sesquilinear form to $\mathcal{H}$ we define the conjugation by

$$v_\mu^\dagger = v_\mu, \qquad u^\dagger = u, \qquad \tilde{u}^\dagger = -\tilde{u}. \tag{2.2}$$

Now we can define in $\mathcal{H}$ the operator Q according to the following formulas:

$$[Q, v_\mu] = i\ \partial_\mu u, \qquad [Q, u] = 0, \qquad [Q, \tilde{u}] = -i\ \partial_\mu v^\mu$$
$$Q\Omega = 0 \tag{2.3}$$

where by $[\cdot, \cdot]$ we mean the graded commutator. One can prove that Q is well defined. Indeed, we have the causal commutation relations

$$[v_\mu(x_1), v_\nu(x_2)] = i\ \eta_{\mu\nu}\ D_0(x_1 - x_2)\ \cdot I, \qquad [u(x_1), \tilde{u}(x_2)] = -i\ D_0(x_1 - x_2)\ \cdot I \tag{2.4}$$

and the other commutators are null. The operator Q should leave invariant these relations, in particular

$$[Q, [v_\mu(x_1), \tilde{u}(x_2)]] + \text{cyclic permutations} = 0 \tag{2.5}$$

which is true according to (2.3). It is useful to introduce a grading in $\mathcal{H}$ as follows: every state which is generated by an even (odd) number of ghost fields and an arbitrary number of vector fields is even (resp. odd). We denote by $|f|$ the grading of the state f. We notice that the operator Q raises the ghost number of a state (of fixed ghost number) by an unit. The usefullness of this construction follows from:

Theorem 2.1 *The operator Q verifies $Q^2 = 0$. The factor space $Ker(Q)/Ran(Q)$ is isomorphic to the Fock space of particles of zero mass and helicity 1 (photons).*

Proof: (i) The fact that Q squares to zero follows easily from (2.3): the operator $Q^2 = 0$ commutes with all field operators and gives zero when acting on the vacuum.

(ii) The generic form of a state $\Psi \in \mathcal{H}^{(1)} \subset \mathcal{H}$ from the one-particle Hilbert subspace is

$$\Psi = \left[\int f_\mu(x) v^\mu(x) + \int g_1(x) u(x) + \int g_2(x) \tilde{u}(x)\right] \Omega \tag{2.6}$$

with test functions f_μ, g_1, g_2 verifying the wave equation equation. We impose the condition $\Psi \in Ker(Q) \iff Q\Psi = 0$; we obtain $\partial^\mu f_\mu = 0$ and $g_2 = 0$ i.e. the generic element $\Psi \in \mathcal{H}^{(1)} \cap Ker(Q)$ is

$$\Psi = \left[\int f_\mu(x) v^\mu(x) + \int g(x) u(x)\right] \Omega \tag{2.7}$$

with g arbitrary and f_μ constrained by the transversality condition $\partial^\mu f_\mu = 0$; so the elements of $\mathcal{H}^{(1)} \cap Ker(Q)$ are in one-one correspondence with couples of test functions (f_μ, g) with the transversality condition on the first entry. Now, a generic element $\Psi' \in \mathcal{H}^{(1)} \cap Ran(Q)$ has the form

$$\Psi' = Q\Phi = \left[\int \partial_\mu g'(x) v^\mu(x) - \int \partial^\mu f'_\mu(x) u(x)\right] \Omega \tag{2.8}$$

so if $\Psi \in \mathcal{H}^{(1)} \cap Ker(Q)$ is indexed by the couple (f_μ, g) then $\Psi + \Psi'$ is indexed by the couple $(f_\mu + \partial_\mu g', g - \partial^\mu f'_\mu)$. If we take f'_μ conveniently we can make $g = 0$. We introduce the equivalence relation $f^{(1)}_\mu \sim f^{(2)}_\mu \quad \Longleftrightarrow f^{(1)}_\mu - f^{(2)}_\mu = \partial_\mu g'$ and it follows that the equivalence classes from $(\mathcal{H}^{(1)} \cap Ker(Q))/(\mathcal{H}^{(1)} \cap Ran(Q))$ are indexed by equivalence classes of wave functions $[f_\mu]$; it remains to prove that the sesquilinear form $< \cdot, \cdot >$ induces a positively defined form on $(\mathcal{H}^{(1)} \cap Ker(Q))/(\mathcal{H}^{(1)} \cap Ran(Q))$ and we have obtained the usual one-particle Hilbert space for the photon.

(iii) We go now to the 2-particle space. We borrow an argument from the proof of Künneth formula [2]. Any 2-particle state is generated by states of the form:

$$\Psi = \sum_{j=1}^{n} f_j \otimes g_j \tag{2.9}$$

with f_j, g_j one-particle states. We impose the condition $\Psi \in Ker(Q)$ and observe that it is sufficient to take f_j, g_j states of fixed ghost number. Moreover, we can take f_j such that their span does not intersect $Ran(Q)$. Indeed if we have constants β_j not all null such that $\sum_{j=1}^{n} \beta_j\ f_j \in Ran(Q)$ then by a redefinition of the vectors f_j we can arrange such that $f_1 = \sum_{j=2}^{n} \beta'_j\ f_j + Qh$. We substitute this in the formula for Ψ and get: $\Psi = \sum_{j=2}^{n} f_j \otimes (\beta'_j g_1 + g_j) + Q(h \otimes g_1) - (-1)^{|h|}\ h \otimes Qg_1$ so if we eliminate the co-boundary we can replace the state Ψ by an equivalent one in which $f_1 \to h$. In this way we replace the expression (2.9) by an equivalent expression for which $\sum_{j=1}^{n} |f_j|$ decreases by an unit. Recursively we obtain another expression (2.9) modulo $Ran(Q)$ for which $Span\ (f_j)_{j=1}^{n} \cap Ran(Q) = \{0\}$. Now the condition $Q\Psi = 0$ writes $\sum_{j=1}^{n} (Qf_j \otimes g_j + (-1)^{|f_j|}\ f_j \otimes Qg_j) = 0$ and it easily follows that both sums must be separately null i.e. we must have $Qg_j = 0$ and $Qf_j = 0$ for all $j = 1, \ldots, n$. It means that we have the canonical isomorphism $(\mathcal{H}^{(2)} \cap Ker(Q))/(\mathcal{H}^{(2)} \cap Ran(Q)) \cong (\mathcal{H}^{(1)} \cap Ker(Q))/(\mathcal{H}^{(1)} \cap Ran(Q)) \otimes (\mathcal{H}^{(1)} \cap Ker(Q))/(\mathcal{H}^{(1)} \cap Ran(Q))$.

Now we can proceed by induction to the general n-particle states. ■

We see that the condition $[Q, T] = i\ \partial_\mu T^\mu$ means that the expression T leaves invariant the physical Hilbert space (at least in the adiabatic limit).

We now consider that the Fock space is generated by a Dirac field ψ of mass m and Fermi statistics. We must supplement (2.1) by

$$< \Omega, \psi(x)\bar{\psi}(y)\Omega >= -i\ S^{(+)}_m(x - y)\ \cdot I, \qquad < \Omega, \bar{\psi}(x)\psi(y)\Omega >= -i\ S^{(-)}_m(y - x)\ \cdot I \tag{2.10}$$

where $S^{(\pm)}_m$ are the corresponding positive (negative) frequency parts of the Dirac causal commutator $S^{(\pm)}_m \equiv i\ (\gamma^\mu \partial_\mu + m) D^{(\pm)}_m$.

Then we have the following result which describes the most general interaction for a zero-mass vector particles and a Dirac field.

Theorem 2.2 *Let T be a relative cocycle for d_Q of canonical dimension $\omega(T) \leq 4$ of ghost number $gh(T) = 0$ and at least tri-linear in the fields. Then T is (relatively) cohomologous to a non-trivial co-cycle of the form:*

$$T = j^\mu\ v_\mu \tag{2.11}$$

where the expression j^μ is bilinear in the Fermi matter fields:

$$j^\mu = e_V\ \overline{\psi}\gamma^\mu\psi + e_A\ \overline{\psi}\gamma^\mu\gamma_5\psi \tag{2.12}$$

and we must have the conservation of the current

$$\partial_\mu j^\mu = 0; \tag{2.13}$$

in particular we can have $e_A \neq 0$ only if the mass of the Dirac field is null.

(ii) The relation $d_Q T = i\ \partial_\mu T^\mu$ is verified by:

$$T^\mu = j^\mu\ u \tag{2.14}$$

and we have $d_Q T^\mu = 0$.

3 General Gauge Theories

We give here the essential ingredients of perturbation theory.

3.1 Bogoliubov Axioms

We first consider arbitrary self-adjoint Wick monomials $W_1, \ldots, W_n$. The chronological products $T(W_1(x_1), \ldots, W_n(x_n)) \quad n = 1, 2, \ldots$ are verifying the following set of axioms:

- Skew-symmetry in all arguments $W_1(x_1), \ldots, W_n(x_n)$:

$$T(\ldots, W_i(x_i), W_{i+1}(x_{i+1}), \ldots,) = (-1)^{f_i f_{i+1}} T(\ldots, W_{i+1}(x_{i+1}), W_i(x_i), \ldots) \tag{3.1}$$

  where f_i is the number of Fermi fields appearing in the Wick monomial W_i.

- Poincaré invariance: for all $(a, A) \in inSL(2, \mathbb{C})$ we have:

$$U_{a,A} T(W_1(x_1), \ldots, W_n(x_n)) U_{a,A}^{-1} = T(A \cdot W_1(A \cdot x_1 + a), \ldots, A \cdot W_n(A \cdot x_n + a)); \tag{3.2}$$

  Sometimes it is possible to supplement this axiom by other invariance properties: space and/or time inversion, charge conjugation invariance, global symmetry invariance with respect to some internal symmetry group, supersymmetry, etc.

- Causality: if $x_i \geq x_j, \quad \forall i \leq k, \quad j \geq k+1$ then we have:

$$T(W_1(x_1), \ldots, W_n(x_n)) = T(W_1(x_1), \ldots, W_k(x_k)) \;\; T(W_{k+1}(x_{k+1}), \ldots, W_n(x_n)); \tag{3.3}$$

- Unitarity: We define the *anti-chronological products* according to

$$(-1)^n \bar{T}(W_1(x_1), \ldots, W_n(x_n)) \equiv \sum_{r=1}^{n} (-1)^r \sum_{I_1, \ldots, I_r \in Part(\{1, \ldots, n\})} \epsilon \;\; T_{I_1}(X_1) \cdots T_{I_r}(X_r) \tag{3.4}$$

  where the we have used the notation:

$$T_{\{i_1, \ldots, i_k\}}(x_{i_1}, \ldots, x_{i_k}) \equiv T(W_{i_1}(x_{i_1}), \ldots, W_{i_k}(x_{i_k})) \tag{3.5}$$

  and the sign ϵ counts the permutations of the Fermi factors. Then the unitarity axiom is:

$$\bar{T}(W_1(x_1), \ldots, W_n(x_n)) = T(W_1(x_1), \ldots, W_n(x_n))^\dagger \tag{3.6}$$

- The "initial condition"

$$T(W(x)) = W(x). \tag{3.7}$$

It can be proved that this system of axioms can be supplemented with

$$\begin{aligned} & T(W_1(x_1), \ldots, W_n(x_n)) \\ = \sum \epsilon \quad < \Omega, T(W_1'(x_1), \ldots, W_n'(x_n)) \Omega > \quad & : W_1''(x_1), \ldots, W_n''(x_n) : \end{aligned} \tag{3.8}$$

where W_i' and W_i'' are Wick submonomials of W_i such that $W_i =: W_i' W_i''$: the sign ϵ counts the number of the permutation of the Fermi fields and Ω is the vacuum state. This is called the *Wick expansion property.*

We can also include in the induction hypothesis a limitation on the order of singularity of the vacuum averages of the chronological products associated to arbitrary Wick monomials $W_1, \ldots, W_n$; explicitly:

$$\omega(< \Omega, T^{W_1, \ldots, W_n}(X) \Omega >) \leq \sum_{l=1}^{n} \omega(W_l) - 4(n-1) \tag{3.9}$$

where by $\omega(d)$ we mean the order of singularity of the (numerical) distribution d and by $\omega(W)$ we mean the canonical dimension of the Wick monomial W; in particular this means that we have

$$T(W_1(x_1),\ldots,W_n(x_n)) = \sum_g t_g(x_1,\ldots,x_n)\; W_g(x_1,\ldots,x_n) \tag{3.10}$$

where W_g are Wick polynomials of fixed canonical dimension and t_g are distributions with the order of singularity bounded by the power counting theorem [5]:

$$\omega(t_g) + \omega(W_g) \leq \sum_{j=1}^{n} \omega(W_j) - 4(n-1) \tag{3.11}$$

and the sum over g is essentially a sum over Feynman graphs.

Up to now, we have defined the chronological products only for self-adjoint Wick monomials $W_1,\ldots,W_n$ but we can extend the definition for arbitrary Wick polynomials by linearity.

One can modify the chronological products without destroying the basic property of causality *iff* one can make

$$\begin{aligned} T(W_1(x_1),\ldots,W_n(x_n)) \rightarrow & \; T(W_1(x_1),\ldots,W_n(x_n)) \\ & + R_{W_1,\ldots,W_n}(x_1,\ldots,x_n) \end{aligned} \tag{3.12}$$

where R are quasi-local expressions; by a *quasi-local expression* we mean an expression of the form

$$R_{W_1,\ldots,W_n}(x_1,\ldots,x_n) = \sum_g \left[P_g(\partial)\delta(X)\right] W_g(x_1,\ldots,x_n) \tag{3.13}$$

with P_g monomials in the partial derivatives and W_g are Wick polynomials; here $\delta(X)$ is the n-dimensional delta distribution $\delta(X) \equiv \delta(x_1-x_n)\cdots\delta(x_{n-1}-x_n)$. Because of the delta function we can consider that P_g is a monomial only in the derivatives with respect to, say $x_2,\ldots,x_n$. If we want to preserve (3.9) we impose the restriction

$$deg(P_g) + \omega(W_g) \leq \sum_{j=1}^{n} \omega(W_j) - 4(n-1) \tag{3.14}$$

and some other restrictions are following from the preservation of Lorentz covariance and unitarity.

The redefinitions of the type (3.12) are the so-called *finite renormalizations*. Let us note that this arbitrariness, described by the number of independent coefficients of the polynomials P_g can grow with n and in this case the theory is called *non-renormalizable*. This can happen if some of the Wick monomials $W_j, j=1,\ldots,n$ have canonical dimension greater than 4. If all the monomials have canonical dimension less of equal to 4 then the arbitrariness is bounded independently of n and the theory is called *renormalizable*.

It is not hard to prove that any finite renormalization can be rewritten in the form

$$R(x_1,\ldots,x_n) = \delta(X)\; W(x_1) + \sum_{j=1}^{n} \frac{\partial}{\partial x_l^{\mu}} R_l(X) \tag{3.15}$$

where the expressions $R_l(X)$ are also quasi-local. But it is clear that the sum in the above expression is null in the adiabatic limit. This means that we can postulate that the finite renormalizations have a much simpler form, namely

$$R(x_1,\ldots,x_n) = \delta(X)\; W(x_1) \tag{3.16}$$

where the Wick polynomial W is constrained by

$$\omega(W) \leq \sum_{j=1}^{n} \omega(W_j) - 4(n-1). \tag{3.17}$$

3.2 Gauge Theories and Anomalies

From now on we consider that we work in the four-dimensional Minkowski space and we have the Wick polynomials T^I such that the descent equations (1.7) are true and we also have

$$[T^I(x),\ T^J(y)] = 0 \tag{3.18}$$

for $x - y$ space-like i.e. these expressions causally commute in the graded sense.

The equation (1.7) are called a *relative cohomology* problem. The co-boundaries for this problem are of the type

$$T^I = d_Q B^I + i\ \partial_\mu B^{I\mu}. \tag{3.19}$$

Next we construct the associated chronological products

$$T^{I_1,\dots,I_n}(x_1,\dots,x_n) = T(T^{I_1}(x_1),\dots,T^{I_n}(x_n)).$$

Because of the previous assumption, it follows from the skew-symmetry axiom that we can choose them such that we have the graded symmetry property:

$$T(\dots,T^{I_k}(x_k),T^{I_{k+1}}(x_{k+1}),\dots) = (-1)^{|I_k||I_{k+1}|}\ T(\dots,T^{I_{k+1}}(x_{k+1}),T^{I_k}(x_k),\dots). \tag{3.20}$$

We also have

$$gh(T^{I_1,\dots,I_n}) = \sum_{l=1}^{n} |I_l|. \tag{3.21}$$

In the case of a gauge theory there are renormalizations of the type (3.13) which call *trivial*, namely those of the type

$$R^{\cdots}(X) = d_Q B^{\cdots}(X) + i\ \sum_{l=1}^{n}\ \frac{\partial}{\partial x^\mu_l} B^{l;\cdots}(X) \tag{3.22}$$

Indeed, as it was remarked above, any co-boundary operator induces the null operator on the physical Hilbert space. Also any total divergence gives a null contribution in the adiabatic limit.

We now write the gauge invariance condition (1.8) in a compact form. We consider the space $\mathcal{C}_n$ of co-chains of the form $C^{I_1,\dots,I_n}(X)$ which are distribution-valued operators in the Hilbert space with antisymmetry in all indices from every $I_j,\ (j = 1,\dots,n)$ and also verifying:

$$C^{\dots,I_k,I_{k+1},\dots}(\dots,x_k,x_{k+1},\dots) = (-1)^{|I_k||I_{k+1}|} \times C^{\dots,I_{k+1},I_k,\dots}(\dots,x_{k+1},x_k,\dots). \tag{3.23}$$

Then we can define the operator $\delta : \mathcal{C}_n \longrightarrow \mathcal{C}_{n+1}$ according to:

$$(\delta\ C)^{I_1,\dots,I_n} \equiv \sum_{l=1}^{n} (-1)^{s_l} \frac{\partial}{\partial x^\mu_l} C^{I_1,\dots,I_l\mu,\dots,I_n}. \tag{3.24}$$

It is easy to prove that we have:

$$\delta^2 = 0; \tag{3.25}$$

we also note that δ commutes with d_Q. One can now write the equation (1.8) in a more compact way:

$$d_Q T^{I_1,\dots,I_n} = i\delta T^{I_1,\dots,I_n}. \tag{3.26}$$

We now determine the obstructions for the gauge invariance relations (3.26). These relations are true for $n = 1$ according to (1.7). If we suppose that they are true up to order $n - 1$ then it follows easily that in order n we must have:

$$d_Q T^{I_1,\dots,I_n} = i\delta T^{I_1,\dots,I_n} + A^{I_1,\dots,I_n} \tag{3.27}$$

where the expressions $A^{I_1,\dots,I_n}(x_1,\dots,x_n)$ are quasi-local operators and are called *anomalies*. It is clear that we have from (3.20) a similar symmetry for the anomalies: namely we have complete antisymmetry in all indices from every $I_j,\ (j = 1,\dots,n)$ and

$$A^{\dots,I_k,I_{k+1},\dots}(\dots,x_k,x_{k+1},\dots) = (-1)^{|I_k||I_{k+1}|} \times A^{\dots,I_{k+1},I_k,\dots}(\dots,x_{k+1},x_k,\dots). \tag{3.28}$$

i.e. $A^{I_1,\dots,I_n}(x_1,\dots,x_n)$ are also co-chains. We also have

$$gh(A^{I_1,\dots,I_n}) = \sum_{l=1}^{n} |I_l| + 1. \tag{3.29}$$

Let $\omega_0 \equiv \omega(T)$; then one has:

$$A^{I_1,\dots,I_n}(X) = 0 \quad \textit{iff} \quad \sum_{l=1}^{n} |I_l| > n(\omega_0 - 4) + 4 \tag{3.30}$$

We can write a more precise form for the anomalies, namely:

$$A^{I_1,\dots,I_n}(x_1,\dots,x_n) = \sum_{k} \sum_{i_1,\dots,i_k>1} [\partial^{i_1}_{\rho_1} \dots \partial^{i_k}_{\rho_k} \delta(X)] W^{I_1,\dots,I_n;\rho_1,\dots,\rho_k}_{i_1,\dots,i_k}(x_1) \tag{3.31}$$

and in this expression the Wick polynomials $W^{I_1,\dots,I_n;\rho_1,\dots,\rho_k}_{i_1,\dots,i_k}$ are uniquely defined. Now from (3.11) we have

$$\omega(W^{I_1,\dots,I_n;\rho_1,\dots,\rho_k}) \leq n(\omega_0 - 4) + 5 - k \tag{3.32}$$

which gives a bound on k in the previous sum. We also have some consistency conditions on the expressions verified by the anomalies. If one applies the operator d_Q to (3.27) one obtains the so-called *Wess-Zumino consistency conditions*:

$$d_Q A^{I_1,\dots,I_n} = -i\ \delta A^{I_1,\dots,I_n}. \tag{3.33}$$

Let us note that we can suppose, as for the finite renormalizations - see (3.16) that all anomalies which are total divergences are trivial because they spoil gauge invariance with terms which can be made as small as one wished, i.e. we can take the form:

$$A(x_1,\dots,x_n) = \delta(X)\ W(x_1). \tag{3.34}$$

It is however interesting that in some cases one can prove that the anomalies can be put in this form by suitable redefinitions of the chronological products. This is the case of quantum electrodynamics which we will analyze in the next Sections.

Suppose now that we have fixed the gauge invariance (3.26) and we investigate the renormalizability issue i.e. we make the redefinitions

$$T(T^{I_1}(x_1),\dots,T^{I_n}(x_n)) \rightarrow T(T^{I_1}(x_1),\dots,T^{I_n}(x_n)) + R^{I_1,\dots,I_n}(x_1,\dots,x_n) \tag{3.35}$$

where R are quasi-local expressions. As before we have

$$R^{\dots,I_k,I_{k+1},\dots}(\dots,x_k,x_{k+1},\dots) = (-1)^{|I_k||I_{k+1}|} \times R^{\dots,I_{k+1},I_k,\dots}(\dots,x_{k+1},x_k,\dots). \tag{3.36}$$

We also have

$$gh(R^{I_1,\dots,I_n}) = \sum_{l=1}^{n} |I_l|. \tag{3.37}$$

and

$$R^{I_1,\dots,I_n} = 0, \quad \sum_{l=1}^{n} |I_l| > n(\omega_0 - 1) + 4. \tag{3.38}$$

If we want to preserve (1.8) it is clear that the quasi-local operators $R^{I_1,\ldots,I_n}$ should also verify

$$d_Q R^{I_1,\ldots,I_n} = i\ \delta R^{I_1,\ldots,I_n} \tag{3.39}$$

i.e. equations of the type (3.33). In this case we note that we have more structure; according to the previous discussion we can impose the structure (3.13):

$$R^{I_1,\ldots,I_n}(x_1,\ldots,x_n) = \delta(X)\ W^{I_1,\ldots,I_n}(x_1) \tag{3.40}$$

and we obviously have:

$$gh(W^{I_1,\ldots,I_n}) = \sum_{l=1}^{n} |I_l| \tag{3.41}$$

and

$$W^{I_1,\ldots,I_n} = 0, \quad \sum_{l=1}^{n} |I_l| > n(\omega_0 - 1) + 4. \tag{3.42}$$

From (3.39) we obtain after some computations that there are Wick polynomials R^I such that

$$W^{I_1,\ldots,I_n} = (-1)^s\ R^{I_1\cup\ldots\cup I_n}. \tag{3.43}$$

where

$$s \equiv \sum_{k<l\leq n} |I_k||I_l|. \tag{3.44}$$

Moreover, we have

$$gh(R^I) = |I| \tag{3.45}$$

and

$$R^I = 0, \quad |I| > n(\omega_0 - 1) + 4. \tag{3.46}$$

Finally, the following descent equations are true:

$$d_Q R^I = i\ \partial_\mu R^{I\mu} \tag{3.47}$$

and have obtained another relative cohomology problem similar to (1.7). The relative co-boundaries of this problem correspond to the relative co-boundaries from (3.12).

4 The Cohomology of the Operator d_Q

The cohomology of the operator d_Q can be reformulated in the language of classical field theory (with Grassmann variables) paying attention to the fact that we are on the mass shell because the Epstein-Glaser construction is done from the very beginning in a Fock space of some free particles.

From the preceding Section we have the physical justification for solving a cohomology problem namely to determine the cohomology of the operator $d_Q = [Q, \cdot]$ induced by Q in the space of Wick polynomials. We consider that the (classical) fields are $v_\mu, u, \tilde{u}$ of null mass and we consider the set $\mathcal{P}$ of polynomials in these fields and their derivatives. We note that on $\mathcal{P}$ we have a natural grading. We introduce by convenience the notation:

$$B \equiv d_\mu v^\mu \tag{4.1}$$

and define the graded derivation d_Q on $\mathcal{P}$ according to

$$d_Q v_\mu = i d_\mu u, \qquad d_Q u = 0, \qquad d_Q \tilde{u} = -i\ B$$
$$[d_Q, d_\mu] = 0. \tag{4.2}$$

Then one can easily prove that $d_Q^2 = 0$ and the cohomology of this operator is isomorphic to the cohomology of the preceding operator (denoted also by d_Q) and acting in the space of Wick monomials. The operator d_Q raises the grading and the canonical dimension by an unit. To determine the cohomology of d_Q it is convenient to introduce the *field strength*

$$F_{\mu\nu} \equiv d_\mu v_\nu - d_\nu v_\mu = v_{\nu;\mu} - v_{\mu;\nu} \tag{4.3}$$

and observe that

$$\begin{aligned} d_Q F_{\mu\nu} &= 0, \\ d_\nu F^{\mu\nu} &= d^\mu B, \\ F_{\mu\nu;\rho} + F_{\nu\rho;\mu} + F_{\rho\mu;\nu} &= 0; \end{aligned} \tag{4.4}$$

the last relation is called *Bianchi identity*. Next we prove that the tensor

$$F^{(0)}_{\mu\nu;\rho_1,\dots,\rho_n} \equiv F_{\mu\nu;\rho_1,\dots,\rho_n} + \frac{1}{n+2} \sum_{l=1}^{n} [\eta_{\mu\rho_l}\ B_{\rho_1,\dots,\hat{\rho}_l,\dots,\rho_n} - (\mu \leftrightarrow \nu)] \tag{4.5}$$

is traceless in all indices and the expressions $F^{(0)}_{\mu\nu;\rho}$ also verify the Bianchi identities. Now we define

$$g_{\mu_1,\dots,\mu_n} \equiv \frac{1}{n} \sum_{l=1}^{n} v_{\mu_l;\mu_1,\dots,\hat{\mu}_l,\dots,\mu_n} \tag{4.6}$$

which is the completely symmetric part of the derivative $v_{\mu_1;\mu_2,\dots,\mu_n}$ and prove that

$$v_{\mu_1;\mu_2,\dots,\mu_n} = g_{\mu_1,\dots,\mu_n} + \frac{1}{n} \sum_{l=2}^{n} d_{\mu_2} \dots \hat{d}_{\mu_l} \dots d_{\mu_n} F_{\mu_l\mu_1}. \tag{4.7}$$

Finally we define

$$g^{(0)}_{\mu_1,\dots,\mu_n} \equiv g_{\mu_1,\dots,\mu_n} - \frac{2}{n(2n+1)} \sum_{1\leq p<q\leq n} \eta_{\mu_p\mu_q}\ B_{\mu_1,\dots,\hat{\mu}_p,\dots,\hat{\mu}_q,\dots,\mu_n} \tag{4.8}$$

which is completely symmetric and traceless.

We will use repeatedly the Künneth theorem:

Theorem 4.1 *Let $\mathcal{P}$ be a graded space of polynomials and d an operator verifying $d^2 = 0$ and raising the grading by an unit. Let us suppose that $\mathcal{P}$ is generated by two subspaces $\mathcal{P}_1, \mathcal{P}_2$ such that $\mathcal{P}_1 \cap \mathcal{P}_2 = \{0\}$ and $d\mathcal{P}_j \subset \mathcal{P}_j, j = 1, 2$. We define by d_j the restriction of d to $\mathcal{P}_j$. Then there exists the canonical isomorphism $H(d) \cong H(d_1) \times H(d_2)$ of the associated cohomology spaces.*

The proof goes in a similar way to the preceding theorem (see [2]). Now we can prove an important result describing the cohomology of the operator d_Q; we denote by Z_Q and B_Q the cocyles and the co-boundaries of this operator.

Theorem 4.2 *Let $p \in Z_Q$. Then p is cohomologous to a polynomial in u and $F^{(0)}_{\mu\nu;\rho_1,\dots,\rho_n}$. If we factorize the space $\mathcal{P}_0 \subset \mathcal{P}$ of such polynomials to the Bianchi identities we obtain a space which is isomorphic to the cohomology space H_Q of d_Q.*

Proof: (i) The idea is to define conveniently two subspaces $\mathcal{P}_1, \mathcal{P}_2$ and apply Künneth theorem. First we use on $\mathcal{P}$ new variables. We eliminate the variables $v_{\mu_1;\mu_2,\dots,\mu_n}$ $(n \geq 2)$ in terms of $g_{\mu_1,\dots,\mu_n}$ $(n \geq 2)$ and $F_{\mu\nu;\rho_1,\dots,\rho_{n-2}}$ using (4.7). Next we eliminate $F_{\mu\nu;\rho_1,\dots,\rho_{n-2}}$ in terms of $F^{(0)}_{\mu\nu;\rho_1,\dots,\rho_{n-2}}$ and $B_{\rho_1,\dots,\rho_{n-2}}$ using (4.5). Finally we eliminate $g_{\mu_1,\dots,\mu_n}$ $(n \geq 2)$ in terms of $g^{(0)}_{\mu_1,\dots,\mu_n}$ $(n \geq 2)$ and $B_{\mu_1,\dots,\mu_{n-2}}$ according to (4.8).

(ii) Now we can take in Künneth theorem $\mathcal{P}_1 = \mathcal{P}_0$ from the statement and $\mathcal{P}_2$ the subspace generated by the variables $B_{\mu_1,\dots,\mu_n}$ $(n \geq 0)$, $g^{(0)}_{\mu_1,\dots,\mu_n}$ $(n \geq 2)$, $\tilde{u}_{\mu_1,\dots,\mu_n}$ $(n \geq 0)$, $u_{\mu_1,\dots,\mu_n}(n > 0)$ and v_μ. We have $d_Q\mathcal{P}_1 = \{0\}$ and

$$\begin{aligned} d_Q u_{\mu_1,\dots,\mu_n} &= 0 \\ d_Q g^{(0)}_{\mu_1,\dots,\mu_n} &= i\ u_{\mu_1,\dots,\mu_n} \\ d_Q \tilde{u}_{\mu_1,\dots,\mu_n} &= -i\ B_{\mu_1,\dots,\mu_n} \\ d_Q B_{\mu_1,\dots,\mu_n} &= 0 \\ d_Q v_\mu &= i u_\mu \end{aligned} \tag{4.9}$$

so we meet the conditions of Künneth theorem. Let us define in $\mathcal{P}_2$ the graded derivation h by:

$$\begin{aligned} h u_\mu &= -i\ v_\mu \\ h u_{\mu_1,\dots,\mu_n} &= -i\ g^{(0)}_{\mu_1,\dots,\mu_n}\ (n \geq 2) \\ h B_{\mu_1,\dots,\mu_n} &= i\ \tilde{u}_{\mu_1,\dots,\mu_n}\ (n \geq 0) \end{aligned} \tag{4.10}$$

and zero on the other variables from $\mathcal{P}_2$. It is easy to prove that h is well defined: the condition of tracelessness is essential to avoid conflict with the equations of motion. Then one can prove that

$$[d_Q, h] = Id \tag{4.11}$$

on polynomials of degree one in the fields and because the left hand side is a derivation operator we have

$$[d_Q, h] = n \cdot Id \tag{4.12}$$

on polynomials of degree n in the fields. It means that h is a homotopy for d_Q restricted to $\mathcal{P}_2$ so the the corresponding cohomology is trivial: indeed, if $p \in \mathcal{P}_2$ is a cocycle of degree n in the fields then it is a coboundary $p = \frac{1}{n} d_Q h p$.

According to Künneth formula if p is an arbitrary cocycle from $\mathcal{P}$ it can be replaced by a cohomologous polynomial from $\mathcal{P}_0$; The description of H_Q follows from $\mathcal{P}_0 \cap B_Q = \emptyset$ and this proves the theorem. ■

5 The Gauge Invariance of Quantum Electrodynamics in All Orders

In quantum electrodynamics we need only a particular form of (3.27) and (3.33) namely the case when we have the canonical dimension $\omega_0 = 4$. In this case (3.30) becomes:

$$A^{I_1,\dots,I_n}(X) = 0 \quad \textit{iff} \quad \sum_{l=1}^{n} |I_l| > 4 \tag{5.1}$$

and this means that only a finite number of the equations (3.27) can be anomalous. We have from (3.27) the following anomalous gauge equations:

$$d_Q T(T(x_1),\dots,T(x_n)) = i \sum_{l=1}^{n} \frac{\partial}{\partial x_l^\mu} T(T(x_1),\dots,T^\mu(x_l),\dots,T(x_n)) + A(X) \tag{5.2}$$

$$d_Q T(T^\mu(x_1),T(x_2),\dots,T(x_n)) = -i \sum_{l=2}^{n} \frac{\partial}{\partial x_l^\nu} T(T^\mu(x_1),T(x_2),\dots,T^\nu(x_l),\dots,T(x_n)) + A^\mu(X) \tag{5.3}$$

$$
\begin{aligned}
& d_Q T(T^\mu(x_1), T^\nu(x_2), T(x_3), \dots, T(x_n)) \\
&= i \sum_{l=3}^n \frac{\partial}{\partial x_l^\rho} T(T^\mu(x_1), T^\nu(x_2), T(x_3), \dots, T^\rho(x_l), \dots, T(x_n)) + A^{\mu\nu}(X)
\end{aligned}
\tag{5.4}
$$

$$
\begin{aligned}
& d_Q T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T(x_4), \dots, T(x_n)) \\
&= -i \sum_{l=4}^n \frac{\partial}{\partial x_l^\sigma} T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T(x_4), \dots, T^\sigma(x_l), \dots, T(x_n)) \\
&\quad + A^{\mu\nu\rho}(X)
\end{aligned}
\tag{5.5}
$$

$$
\begin{aligned}
& d_Q T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T^\sigma(x_4), \dots, T(x_n)) \\
&= i \sum_{l=5}^n \frac{\partial}{\partial x_l^\lambda} T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T^\sigma(x_4), T(x_5), \dots, T^\lambda(x_l), \dots, T(x_n)) \\
&\quad + A^{\mu\nu\rho\sigma}(X)
\end{aligned}
\tag{5.6}
$$

where we can assume, without losing generality, that:

$$
\begin{aligned}
A^{\mu\nu}(X) &= 0, \quad |X| = 1, \\
A^{\mu\nu\rho}(X) &= 0, \quad |X| \leq 2, \\
A^{\mu\nu\rho\sigma}(X) &= 0, \quad |X| \leq 3.
\end{aligned}
\tag{5.7}
$$

From (3.28), we get the following symmetry properties:

$$
A(x_1, \dots, x_n) \quad \text{is symmetric in} \quad x_1, \dots, x_n; \tag{5.8}
$$

$$
A^\mu(x_1, \dots, x_n) \quad \text{is symmetric in} \quad x_2, \dots, x_n; \tag{5.9}
$$

$$
A^{\mu\nu}(x_1, \dots, x_n) \quad \text{is symmetric in} \quad x_3, \dots, x_n; \tag{5.10}
$$

$$
A^{\mu\nu\rho}(x_1, \dots, x_n) \quad \text{is symmetric in} \quad x_4, \dots, x_n; \tag{5.11}
$$

$$
A^{\mu\nu\rho\sigma}(x_1, \dots, x_n) \quad \text{is symmetric in} \quad x_5, \dots, x_n \tag{5.12}
$$

and we also have:

$$
A^{\mu\nu}(x_1, \dots, x_n) = -A^{\nu\mu}(x_2, x_1, x_3, \dots, x_n); \tag{5.13}
$$

$$
A^{\mu\nu\rho}(x_1, \dots, x_n) = -A^{\nu\mu\rho}(x_2, x_1, x_3, \dots, x_n) = -A^{\mu\rho\nu}(x_1, x_3, x_2, x_4, \dots, x_n); \tag{5.14}
$$

$$
\begin{aligned}
& A^{\mu\nu\rho\sigma}(x_1, \dots, x_n) = -A^{\nu\mu\rho\sigma}(x_2, x_1, x_3, \dots, x_n) \\
&= -A^{\mu\rho\nu\sigma}(x_1, x_3, x_2, x_4, \dots, x_n) = -A^{\mu\nu\sigma\rho}(x_1, x_2, x_4, x_3, x_5, \dots, x_n).
\end{aligned}
\tag{5.15}
$$

The Wess-Zumino consistency conditions are in this case:

$$
d_Q A(x_1, \dots, x_n) = -i \sum_{l=1}^n \frac{\partial}{\partial x_l^\mu} A^\mu(x_l, x_1, \dots, \hat{x}_l, \dots, x_n) \tag{5.16}
$$

$$
d_Q A^\mu(x_1, \dots, x_n) = i \sum_{l=2}^n \frac{\partial}{\partial x_l^\nu} A^{\mu\nu}(x_1, x_l, x_2, \dots, \hat{x}_l, \dots, x_n) \tag{5.17}
$$

$$
d_Q A^{\mu\nu}(x_1, \dots, x_n) = -i \sum_{l=3}^n \frac{\partial}{\partial x_l^\rho} A^{\mu\nu\rho}(x_1, x_2, x_l, x_3, \dots, \hat{x}_l, \dots, x_n) \tag{5.18}
$$

$$
d_Q A^{\mu\nu\rho}(x_1, \dots, x_n) = i \sum_{l=4}^n \frac{\partial}{\partial x_l^\rho} A^{\mu\nu\rho\sigma}(x_1, x_2, x_3, x_l, x_4, \dots, \hat{x}_l, \dots, x_n) \tag{5.19}
$$

$$
d_Q A^{\mu\nu\rho\sigma}(x_1, \dots, x_n) = 0. \tag{5.20}
$$

We recall that the generic form of the anomalies is given by (3.31). We propose to simplify this expression using appropriate redefinitions of the chronological products.

Theorem 5.1 *One can redefine the chronological products such that*

$$A(X) = \delta(X)\; W(x_1) \qquad A^I = 0,\; I \neq 0 \tag{5.21}$$

where the Wick polynomial W has the generic form:

$$W = c_1\; u + c_2\; u\; \overline{\psi}\psi + c_3\; u\; \overline{\psi}\gamma_5\psi + c_4\; u\; F^{\mu\nu}\; F_{\mu\nu} + c_5\; \epsilon_{\mu\nu\rho\sigma}\; u\; F^{\mu\nu}\; F^{\rho\sigma}. \tag{5.22}$$

Proof: (i) We start with $A^{\mu\nu\rho\sigma}$ which has ghost number 5. From (3.31) we have

$$A^{\mu\nu\rho\sigma}(X) = \delta(X)\; W^{\mu\nu\rho\sigma}(x_1) \tag{5.23}$$

and the equation (5.20) gives

$$d_Q\; W^{\mu\nu\rho\sigma} = 0. \tag{5.24}$$

We apply theorem 4.2 and it follows that

$$W^{\mu\nu\rho\sigma} = d_Q B^{\mu\nu\rho\sigma} + W_0^{\mu\nu\rho\sigma} \tag{5.25}$$

where $W_0^{\mu\nu\rho\sigma} \in \mathcal{P}_0$. But it is easy to prove that there are no such expressions in canonical dimension and ghost number 5 so in fact $W^{\mu\nu\rho\sigma}$ is a coboundary. It we redefine the chronological products (i.e. we perform a finite renormalization):

$$\begin{aligned} &T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T^\sigma(x_4), \ldots, T(x_n)) \quad\longrightarrow \\ &T(T^\mu(x_1), T^\nu(x_2), T^\rho(x_3), T^\sigma(x_4), \ldots, T(x_n)) + \delta(X)\; B^{\mu\nu\rho\sigma}(x_1) \end{aligned} \tag{5.26}$$

then we make

$$A^{\mu\nu\rho\sigma}(X) = 0. \tag{5.27}$$

(ii) Now we investigate $A^{\mu\nu\rho}$ which has ghost number 4. The Wess-Zumino equation (5.19) is now:

$$d_Q A^{\mu\nu\rho}(X) = 0 \tag{5.28}$$

and we can proceed as above and eliminate this anomaly.

This goes in the same way for $A^{\mu\nu}$ and A^μ so we are left only with the anomaly A in which only the piece

$$A(X) = \delta(X)\; W(x_1) \tag{5.29}$$

is non-trivial. From the condition $d_Q W = 0$ we have

$$W = d_Q B + W_0 \tag{5.30}$$

where $W_0 \in \mathcal{P}_0$. The coboundary can be eliminated using a finite renormalization and the generic form of $W = W_0 \in \mathcal{P}_0$ is given in the statement. ■

An important observation is the following one. Let us define the so-called *charge conjugation* operator according to

$$\begin{aligned} U_c\; v_\mu\; U_c^{-1} = -v_\mu, \qquad U_c\; u\; U_c^{-1} = -u, \qquad &U_c\; \tilde{u}\; U_c^{-1} = -\tilde{u}, \\ &U_c\; \psi\; U_c^{-1} = -C\; \gamma_0\; \psi^\dagger, \\ &U_c\Omega = 0 \end{aligned} \tag{5.31}$$

where C is the *charge conjugation matrix.* Then we can easily prove that we have

$$U_c\; T\; U_c^{-1} = T, \qquad U_c\; T^\mu\; U_c^{-1} = T^\mu. \tag{5.32}$$

Then we have the main result:

Theorem 5.2 *The chronological products can be chosen such that the theory is gauge invariant in all orders of perturbation theory.*

Proof: (i) First we can define the chronological products such that they are charge conjugation invariant in all orders of perturbation theory by induction. We suppose that the assertion is true up to order $n-1$ i.e.

$$U_c\ T^{I_1,\dots,I_k}\ U_c^{-1} = T^{I_1,\dots,I_k}, \qquad k < n.$$

If $T^{I_1,\dots,I_n}$ do not verify this relation we simply replace:

$$T^{I_1,\dots,I_n} \rightarrow \frac{1}{2}\ (T^{I_1,\dots,I_n} + U_c\ T^{I_1,\dots,I_n}\ U_c^{-1}). \tag{5.33}$$

So we can suppose that we have

$$U_c\ T^{I_1,\dots,I_k}\ U_c^{-1} = T^{I_1,\dots,I_k}, \qquad \forall n. \tag{5.34}$$

(ii) Suppose now that the theory is gauge invariant up to order $n-1$. Then in order n we might have the anomalies $A^{I_1,\dots,I_k}$. From the preceding relation and (3.27) we now have

$$U_c\ A^{I_1,\dots,I_k}\ U_c^{-1} = A^{I_1,\dots,I_k}, \qquad \forall n. \tag{5.35}$$

Using the preceding theorem it follows:

$$U_c\ W\ U_c^{-1} = W. \tag{5.36}$$

If we substitute the generic expression (5.22) in the preceding relation we obtain $W = 0$ and this proves gauge invariance in order n. ■

We emphasize that the main property used in the proof was charge conjugation invariance. This idea goes back to the so-called Furry theorem. In the similar way one can treat other models for which a charge conjugation operator do exists e.g. scalar electrodynamis and $SU(n)$ invariant models without spontaneously broken symmetry.

6 Conclusions

The cohomological methods presented in this paper leads to the most simple understanding of quantum gauge models in perturbation theory and extract completely the information from the consistency Wess-Zumino equations. We have illustrate the methods for the case of quantum electrodynamics. The same methods work for the case of general Yang-Mills models [11] and quantum gravity considered as a perturbative theory of particles of helicity (spin) 2. However in these cases we do not have in general charge conjugation invariance so the elimination of anomalies in all orders using some Furry type argument is not possible. However a lot of interesting informations can be obtained from the elimination of the anomalies in lower orders of the perturbation theory for which explicit computations are possible.

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