# The $2 \nu \beta \beta$ decay $0^{+} \rightarrow 2^{+}$within a boson expansion approach 

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#### Abstract

The Gamow-Teller transition operator is written as a polynomial in the dipole proton-neutron and quadrupole charge conserving QRPA boson operators, using the prescription of the boson expansion technique. Then, the $2 \nu \beta \beta$ process ending on the first $2^{+}$state in the daughter nucleus is allowed through one, two and three boson states describing the odd-odd intermediate nucleus. The approach uses a single particle basis which is obtained by projecting out the good angular momentum from an orthogonal set of deformed functions. The basis for mother and daughter nuclei may have different deformations. The GT transition amplitude as well as the half lives were calculated for eighteen transitions. Results are compared with the available data as well as with the predictions obtained with other methods.


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## 1 Introduction

One of the most exciting subjects of nuclear physics is that of double beta decay. The interest is generated by the fact that in order to describe quantitatively the decay rate one has to treat consistently the neutrino properties as well as the nuclear structure features. The process may take place in two distinct ways: a) by a $2 \nu \beta \beta$ decay where the initial nuclear system, the mother nucleus, is transformed in the final stable nuclear system, usually called the daughter nucleus, two electrons and two anti-neutrinos; b) by the $0 \nu \beta \beta$ process where the final state does not involve any neutrino. The first process preserves the lepton number while the second one does not.

The latter decay mode is especially interesting since one hopes that its discovery might provide a definite answer to the question whether the neutrino is a Majorana or a Dirac particle. I recall you that the Dirac particle is different from its antiparticle while the Majorana particle coincides with the corresponding antiparticle. The problem concerning the nature of neutrino is a long standing issue. Indeed, even in 1955 Davis [1] had the idea to place in a reactor ${ }^{37} \mathrm{Cl}$. Inside reactor there are plenty of neutrons resulting from fission processes. Part of these particle dezintegrate through the $\beta$ process:

$$
\begin{equation*}
n \rightarrow p+e^{-}+\tilde{\nu} \tag{1.1}
\end{equation*}
$$

| $\nu_{L}$ | $d_{y}^{c}$ | $d_{r}^{c}$ | $d_{b}^{c}$ | $u_{b}$ | $u_{r}$ | $u_{y}$ | $e^{+}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $e^{-}$ | $u_{y}^{c}$ | $u_{r}^{c}$ | $u_{b}^{c}$ | $d_{b}$ | $d_{r}$ | $d_{y}$ | $\nu_{R}^{c}$ |

If $\tilde{\nu}$ would be identical with $\nu$ then it produces the inverse reaction

$$
\begin{equation*}
\nu+{ }^{37} \mathrm{Cl} \rightarrow{ }^{37} \mathrm{Ar}+e^{-} \tag{1.2}
\end{equation*}
$$

However the result of the experiment was negative, i.e no atom ${ }^{37} \mathrm{Ar}$ has been recorded. Davis concluded that neutrino is a Dirac particle. This experiment took place one year before the discovery of CP violation [2,3]. With the new knowledge of parity violation one has been realized that the process can be forbidden also in the case the neutrino is a massless Majorana particle, due to the helicity conservation. Indeed, if the $0 \nu \beta \beta$ decay exists then the antineutrino emitted by the first neutron should be absorbed by the second neutron at a latter time. The emitted antineutrino has however a right helicity while the absorbed neutrino in the second process must have a left helicity. Therefore the process is forbidden. Therefore the dilemma about whether neutrino is a Dirac or Majorana particle still persists. We notice that the process would be allowed if some mechanisms of flipping the helicity are active. Indeed, the massive neutrino could flip its helicity due to its interaction with an electroweak right handed current.

Why such an old issue is still intensively studied nowadays? The reason is that some formalisms forbid the process of $0 \nu \beta \beta$ while other ones allow it $[4,5,6]$. Indeed, since the process does not preserve the lepton number, it is forbidden by gauge theories like $S U(5)$ and $S U(2) \times U(1)$. On the other hand there are GUTs (Grand Unified Theories) which predict a right handed neutrino and therefore the existence of the $0 \nu \beta \beta$ process. Indeed, the GUT formalism based on $S O(10)$ symmetry predicts a family of 16 fermions presented in Table I. The first 4 fermions from the first row and $e^{-}$constitute a 5 -dimensional representation of $S U(5)$, the $\nu_{R}^{c}$ is a one dimensional while the remaining fermions formed out of the quarks up and down with the colors yellow, red and blue belong to a 10-dimensional representation of $S U(5)$. In this model the neutrino has both a Dirac and a Majorana mass. The mass is represented by a matrix which in the case of only one flavor is a $2 \times 2$ matrix which yields by diagonalization:

$$
\begin{gather*}
\nu_{1} \approx \nu_{L}-\frac{m^{D}}{m_{R}^{M}} \nu_{R}, \quad m_{1} \approx \frac{\left(m^{D}\right)^{2}}{M_{R}^{M}} \\
\nu_{2} \approx \nu_{R}+\frac{m^{D}}{m_{R}^{M}} \nu_{L}, \quad m_{2}=M_{R}^{M} \tag{1.3}
\end{gather*}
$$

The Dirac and the right Majorana mass are $m^{D} \approx 1 . M e V, \quad m_{R}^{M} \geq 10^{3} \mathrm{GeV}$. From these data it results $m_{1} \leq 1 . e V$. The weak point of this formalism is that there is no left-handed Majorana mass. However it is shown how one generates Majorana neutrino masses by mixing a large Dirac neutrino mass with an even larger Majorana neutrino mass terms.

The conclusion is that the GUT's predict a mass for neutrino and a coupling to a right handed field, i.e. a right handed neutrino $\nu_{R}$. In this context one may assert that the study of the $0 \nu \beta \beta$ decay could confirm or reject the predictions of the GUT's formalism. Actually this is the reason why the issue of finding the nature of neutrino is considered to be of a paramount importance.

There are two sources for experimental information on $\beta \beta$ decay: a) the direct counting technique and b) the geochemical technique. The laboratory counting experiments allow to separate the $2 \nu$ and $0 \nu$ modes. On the other hand the geochemical measurements determine the total abundance of the final nucleus, giving therefore the total decay rate and a lower limit on the two modes partial life times. Using these data and the nuclear matrix elements one may calculate the upper limits for both the effective neutrino mass and the effective right handedness of its interaction.

The drawbacks of these predictions is that no reliable direct test for the nuclear matrix elements, exists. Fortunately, similar nuclear matrix elements are involved also in calculations for the $2 \nu \beta \beta$ process for which plenty of data exist. Therefore, a good idea would be to use for $0 \nu \beta \beta$ decay those matrix elements which realistically describe the decay rate for the $2 \nu \beta \beta$ process.

To conclude, the $0 \nu \beta \beta$ decay exhibits a double fold importance: a) provides information about the neutrino nature b) yields important restrictions of GUT's.

I shall restrict my considerations to the $2 \nu \beta \beta$ decay. This process is interesting by its own, but is very attractive because it supplies us with realistic matrix elements which can be used in studying the neutrinoless double beta decay.

The contributions over several decades have been reviewed by many authors. [7, 8, 9, 10, 11, 12].
In what follows, I shall mention the features which have been treated by my group in a series of publications.

Although none of the double beta emitters is a spherical nucleus most formalisms use a single particle spherical basis.

In the middle of 90 's we treated the $2 \nu \beta \beta$ process in a pnQRPA formalism using a projected spherical single particle basis which resulted in having a unified description of the process for spherical and deformed nuclei $[13,14]$. Recently the single particle basis [ 15,16$]$ has been improved by accounting for the volume conservation while the mean field is deformed [17, 18]. The improved basis has been used for describing quantitatively the ground state to ground state double beta decay rates as well as the corresponding half lives [19, 20]. The results were compared with the available data as well as with the predictions of other formalisms. The manners in which the physical observable is influenced by the nuclear deformations of mother and daughter nuclei are in detail commented. Two features of the deformed basis are essential: a) the single particle energy levels do not exhibit any gap; b) the pairing properties of the deformed system are different from those of spherical system. These two aspects of the deformed nuclei affect the overlap matrix of the pnQRPA states of mother and daughter nuclei. Moreover, considering the Gamow-Teller (GT) transition operator in the single particle-space generated by the deformed mean-field, one obtains an inherent renormalization with respect to the one acting in a spherical basis.

In Ref. [21] we studied the higher pnQRPA effects on the GT transition amplitude, by means of the boson expansion technique for a spherical single particle basis. Considering higher order boson expansion terms in the transition operator, significant corrections to the GT transition amplitude are obtained especially when the strength of the two body particle-particle ( $p p$ ) interaction approaches its critical value where the lowest dipole energy is vanishing. As we showed in the quoted reference, there are transitions which are forbidden at the pnQRPA level but allowed once the higher pnQRPA corrections are included. An example of this type is the $2 \nu \beta \beta$ decay leaving the daughter nucleus in a collective excited state $2^{+}$. The electrons resulting in this process can be distinguished from the ones associated to the ground to ground transition by measuring, in coincidence, the gamma rays due to the transition $2^{+} \rightarrow 0^{+}$in the daughter nucleus [23].

As specified by the title, my talk will refer to the study the $2 \nu \beta \beta$ decay $0^{+} \rightarrow 2^{+}$, where $0^{+}$is the ground state of the emitter while $2^{+}$is a single quadrupole phonon state describing the daughter nucleus. The adopted procedure is the boson expansion method as formulated in our previous paper [21], but using a projected spherical single particle basis. It is worth mentioning that despite the fact the boson expansion approach has been widely used for two alike fermion operators, the procedure has been extended for proton-neutron operators only in the beginning of 90 's [21] for spherical single particle basis and recently for a deformed mean field [22].

Concluding, our formalism involves two basic ingredients: a) the boson expansion approach for the Gamow-Teller transition operator. In this way the $2 \nu \beta \beta$ decay $0^{+} \rightarrow 2^{+}$, forbidden in the framework of the pnQRPA formalism, becomes an allowed process; b) a projected spherical single particle basis. Using such a basis one may unitarily treat the transitions of spherical and deformed nuclei. Moreover, situations when the mother and daughter nuclei have different nuclear deformations could be accounted for.

## 2 The Gamow-Teller transition amplitude

### 2.1 The $2 \nu \beta \beta$ decay $0^{+} \rightarrow 0^{+}$

In our treatment the Fermi transitions, contributing about $20 \%$ to the total rate, and the "forbidden" transitions are ignored, which is a reasonable approximation for the two neutrino double beta decay in medium and heavy nuclei. Therefore we suppose that the Gamow-Teller transitions are dominant for the above mentioned nuclei. If the energy carried by leptons in the intermediate state is approximated by the sum of the rest energy of the emitted electron and half the Q -value of the ground to ground double beta decay process

$$
\begin{equation*}
\Delta_{00} E=m_{e} c^{2}+\frac{1}{2} Q_{\beta \beta}^{(0 \rightarrow 0)}, \tag{2.1}
\end{equation*}
$$

the reciprocal value of the $2 \nu \beta \beta$ half life can be factorized as:

$$
\begin{equation*}
T_{1 / 2}^{2 \nu}\left(0_{i}^{+} \rightarrow 0_{f}^{+}\right)^{-1}=G_{00}\left|M_{G T}^{(00)}\right|^{2} \tag{2.2}
\end{equation*}
$$

where $G_{00}$ is the Fermi integral which characterizes the phase space of the process while the second factor is the GT transition amplitude tude which, in the second order of perturbation theory, has the expression:

$$
\begin{equation*}
M_{G T}^{(00)}=\sqrt{3} \sum_{k, m} \frac{{ }_{i}\left\langle 0\left\|\beta^{+}\right\| k, m\right\rangle_{i}\left\langle k, m \mid k^{\prime}, m^{\prime}\right\rangle_{f f}\left\langle k^{\prime}, m^{\prime}\left\|\beta^{+}\right\| 0_{1}^{+}\right\rangle_{f}}{E_{k, m}+\Delta E_{1}} . \tag{2.3}
\end{equation*}
$$

Here $\Delta E_{1}=\Delta_{00} E+E_{1^{+}}$, with $E_{1+}$ standing for the experimental energy for the first $1^{+}$.
The two matrix elements involved in Eq.2.3 determine the single $\beta^{-}$and $\beta^{+}$transition strengths, respectively. These are shown in Figs. 1-4 for the nuclei treated in the works presented in the present lecture.

Note that in order to obtain a large amplitude for the double beta transition is necessary that the single beta transitions strength represented in the left and right columns achieve the maximum values for the same pnQRPA energies. Note that due to the repulsive character of the $p h$ interaction the $\beta^{-}$ strength in the quasiparticle representation is pushed towards the higher energies when the pnQRPA correlations are introduced.

### 2.2 The $2 \nu \beta \beta$ decay $0^{+} \rightarrow 2^{+}$

If the final state is the collective state $2^{+}$characterizing the daughter nucleus and under the same approximations, i.e. neglecting the Fermi transitions and replacing the lepton energy of the intermediate state by

$$
\begin{equation*}
\Delta_{02} E=m_{e} c^{2}+\frac{1}{2} Q_{\beta \beta}^{(0 \rightarrow 2)}, \tag{2.4}
\end{equation*}
$$

a similar expression for the process half-life is obtained:

$$
\begin{equation*}
T_{1 / 2}^{2 \nu}\left(0_{i}^{+} \rightarrow 2_{f}^{+}\right)^{-1}=G_{02}\left|M_{G T}^{(02)}\right|^{2}, \tag{2.5}
\end{equation*}
$$

where $G_{02}$ is the Fermi integral which characterizes the phase space of the process while the second factor is the GT transition amplitude :

$$
\begin{equation*}
M_{G T}^{(02)}=\sqrt{3} \sum_{k, m} \frac{i\left\langle 0\left\|\beta^{+}\right\| k, m\right\rangle_{i}\left\langle k, m \mid k^{\prime}, m^{\prime}\right\rangle_{f f}\left\langle k^{\prime}, m^{\prime}\left\|\beta^{+}\right\| 2_{1}^{+}\right\rangle_{f}}{\left(E_{k, m}+\Delta E_{2}\right)^{3}} . \tag{2.6}
\end{equation*}
$$

Here $\Delta E_{2}=\Delta_{02} E+E_{1^{+}}$, with $E_{1^{+}}$standing for the experimental energy for the first state $1^{+}$.
Comparing the expressions 2.3 and 2.6 we notice some differences:i) the Fermi factors are different since they depend on the $Q$-values of the processes as well as on the angular momentum of the final state; ii) the denominator of the transition $0^{+} \rightarrow 2^{+}$has a cubic power while for the transition ground to ground the denominator is linear in the energy of the intermediate state. iii) the final states


Figure 1: The single $\beta^{-}$(left column) and $\beta^{+}$ (right column) transition strengths are plotted as function of pnQRPA energies (continuous curve) or of the dipole two quasiparticle energies (dashed line). The continuous curves have been obtained by folding the data with a gaussian od a width equal to 0.5 MeV . The chosen beta ${ }^{-}$and $\beta^{+}$emitters are ${ }^{48} \mathrm{Ca},{ }^{96} \mathrm{Zr},{ }^{100} \mathrm{Mo}$; ${ }^{48} \mathrm{Ti},{ }^{96} \mathrm{Mo},{ }^{100} \mathrm{Ru}$, respectively.


Figure 2: The single $\beta^{-}$(left column) and $\beta^{+}$ (right column) transition strengths are plotted as function of pnQRPA energies (continuous curve) or of the dipole two quasiparticle energies (dashed line). The continuous curves have been obtained by folding the data with a gaussian od a width equal to 0.5 MeV . The chosen beta ${ }^{-}$and $\beta^{+}$emitters arer ${ }^{104} \mathrm{Ru},{ }^{110} \mathrm{Pd},{ }^{116} \mathrm{Cd}$; ${ }^{104} \mathrm{Pd},{ }^{110} \mathrm{Cd},{ }^{116} \mathrm{Sn}$, respectively.
characterizing the two processes are different. The low indices $i$ and $f$ accompanying the nuclear states suggest that these states describe the initial and final nuclei respectively. The operator $\beta^{+}$is the $\beta^{-}$decay transition operator. Indeed, acting with this operator on the right state, describing the ground state of the even-even mother nucleus $(N, Z)$, i.e. ${ }_{i}\langle 0|$, one obtains a state characterizing the neighboring odd-odd nucleus ( $N-1, Z+1$ ). Its explicit expression will be given in Section ... If this operator acts on the state staying at its right side plays the role of a $\beta^{+}$transition operator. Thus, if that right state is the state $|0\rangle_{f}$, characterizing the daughter nucleus ( $N-2, Z+2$ ), the excited state describes the intermediate odd-odd nucleus $(N-1, Z+1)$. The two sets of states obtained by exciting the mother and daughter nuclei respectively are not orthogonal onto each other. As a matter of fact this feature explains the need of introducing the overlap matrix appearing at the numerator in the transition amplitude.

Concluding, the transition amplitude involves two reduced matrix elements which are associated with the beta ${ }^{-}$and $\beta^{+}$transitions of the mother and daughter nuclei respectively. Alternatively, the second matrix element could be viewed as being associated to the $\beta^{-}$decay of the intermediate oddodd nucleus to the final state describing the daughter nucleus. Therefore the double beta decay may be conceived as consisting of two consecutive single $\beta^{-}$transitions.

In Fig. 5 one sketches a particular double beta transition. Note that the first single beta transition is virtual since the final state is an excited state with respect to the initial state. That means that the life-time of such a state should be so that the uncertainty relation for the energy and time variables be obeyed.

The task of the theoretical methods is to provide a description of the nuclear states characterizing the initial, intermediate and final nuclei, respectively. If the adopted formalism involves the pairing correlations then the two legs of the double beta transition are accompanied by products of $u$ and $v$ coefficients defining the Bogoliubov-Valatin transformation. These factors are decisive for the relative


Figure 3: THe same as in Fig. 1 but for ${ }^{128} \mathrm{Te},{ }^{130} \mathrm{Te},{ }^{134} \mathrm{Xe} ;{ }^{128} \mathrm{Xe},{ }^{130} \mathrm{Xe},{ }^{134} \mathrm{Ba}$.
magnitude of the two matrix elements. Thus the first leg matrix element is large while the second leg matrix element associated to the virtual $\beta^{+}$decay of the daughter nucleus is small. If the pairing correlations are washed out the second matrix element is vanishing. Hence, the process of double beta decay is forbidden within a spherical shell model calculation.

Calculations which yield transition rates for the ground to ground double beta transitions are


Figure 4: The same for Fig. 1 but for ${ }^{136} \mathrm{Xe} ;{ }^{136} \mathrm{Ba}$.


Figure 5: One illustrates the double beta GT transitions $0^{+} \rightarrow 0^{+}$from ${ }^{76} \mathrm{Ge}$ to ${ }^{76} \mathrm{Se}$ via the intermediate odd-odd nucleus which is ${ }^{76} \mathrm{As}$. The coefficients $V^{2}\left(U^{2}\right)$ are the occupation (non-occupation) probability for the state specified by the set of quantum numbers mentioned generically by $p$ and $n$.
based on pnQRPA (proton-neutron quasiparticle random phase approximation) formalism.
Here we shall present a $p n Q R P A$ formalism which uses a projected spherical single particle bases. This basis allows us to describe in an unified fashion the decay properties of both spherical and deformed nuclei. Within the $p n Q R P A$ approach the transition operator is linear in the protonneutron phonon operators. Due to this feature the only intermediate states which contribute to the Gamow-Teller transition amplitude are the one phonon proton-neutron dipole states. However, such a transition operator produces vanishing matrix elements for the second leg of the double beta transition. Therefore the double beta decay $0^{+} \rightarrow 2^{+}$is forbidden within the pnQRPA approach. However, if we add to the transition operator higher $p n Q R P A$ corrections the process will be allowed.

Summarizing, two ingredients are specific to the present approach:i) a projected spherical single particle basis and ii) a boson expansion representation of the single beta transition operator. These to original contributions to the present issue will be described in some details in Sections 3 and 4.

## 3 A projected spherical single particle basis

The single particle mean field is determined by a particle-core Hamiltonian:

$$
\begin{equation*}
\tilde{H}=H_{\text {sm }}+H_{\text {core }}-M \omega_{0}^{2} r^{2} \sum_{\lambda=0,2} \sum_{-\lambda \leq \mu \leq \lambda} \alpha_{\lambda \mu}^{*} Y_{\lambda \mu}, \tag{3.1}
\end{equation*}
$$

where $H_{s m}$ denotes the spherical shell model Hamiltonian while $H_{\text {core }}$ is a harmonic quadrupole boson $\left(b_{\mu}^{+}\right)$Hamiltonian associated to a phenomenological core. The interaction of the two subsystems is accounted for by the third term of the above equation, written in terms of the shape coordinates $\alpha_{00}, \alpha_{2 \mu}$. The quadrupole shape coordinates and the corresponding momenta are related to the quadrupole boson operators by the canonical transformation:

$$
\begin{equation*}
\alpha_{2 \mu}=\frac{1}{k \sqrt{2}}\left(b_{2 \mu}^{\dagger}+(-)^{\mu} b_{2,-\mu}\right), \pi_{2 \mu}=\frac{i k}{\sqrt{2}}\left((-)^{\mu} b_{2,-\mu}^{\dagger}-b_{2 \mu}\right), \tag{3.2}
\end{equation*}
$$

where $k$ is an arbitrary $C$ number. The monopole shape coordinate is determined from the volume conservation condition. In the quantized form, the result is:

$$
\begin{equation*}
\alpha_{00}=\frac{1}{2 k^{2} \sqrt{\pi}}\left[5+\sum_{\mu}\left(2 b_{\mu}^{\dagger} b_{\mu}+\left(b_{\mu}^{\dagger} b_{-\mu}^{\dagger}+b_{-\mu} b_{\mu}\right)(-)^{\mu}\right)\right] . \tag{3.3}
\end{equation*}
$$

Averaging $\tilde{H}$ on the eigenstates of $H_{s m}$, hereafter denoted by $|n l j m\rangle$, one obtains a deformed boson Hamiltonian whose ground state is, in the harmonic limit, described by a coherent state

$$
\begin{equation*}
\Psi_{g}=\exp \left[d\left(b_{20}^{+}-b_{20}\right)\right]|0\rangle_{b}, \tag{3.4}
\end{equation*}
$$

with $|0\rangle_{b}$ standing for the vacuum state of the boson operators and $d$ a real parameter which simulates the nuclear deformation. On the other hand, the average of $\tilde{H}$ on $\Psi_{g}$ is similar to the Nilsson Hamiltonian [24]. Due to these properties, it is expected that the best trial functions to generate a spherical basis are:

$$
\begin{equation*}
\Psi_{n l j}^{p c}=|n l j m\rangle \Psi_{g} . \tag{3.5}
\end{equation*}
$$

The projected states are obtained by acting on these deformed states with the projection operator

$$
\begin{equation*}
P_{M K}^{I}=\frac{2 I+1}{8 \pi^{2}} \int D_{M K}^{I}{ }^{*}(\Omega) \hat{R}(\Omega) d \Omega, \tag{3.6}
\end{equation*}
$$

where $D_{M K}^{I}(\Omega)$ denotes the rotation matrix corresponding to the Euler angles $\Omega$. The subset of projected states :

$$
\begin{equation*}
\Phi_{n l j}^{I M}(d)=\mathcal{N}_{n l j}^{I} P_{M I}^{I}\left[|n l j I\rangle \Psi_{g}\right] \equiv \mathcal{N}_{n l j}^{I} \Psi_{n l j}^{I M}(d), \tag{3.7}
\end{equation*}
$$

are orthogonal with the normalization factor denoted by $\mathcal{N}_{n l j}^{I}$.
Although the projected states are associated to the particle-core system, they can be used as a single particle basis. Indeed, when a matrix element of a particle like operator is calculated, the integration on the core collective coordinates is performed first, which results in obtaining a final factorized expression: one factor carries the dependence on deformation and one is a spherical shell model matrix element.

The single particle energies are approximated by the average of the particle-core Hamiltonian $H^{\prime}=\tilde{H}-H_{\text {core }}$ on the projected spherical states defined by Eq.(3.7):

$$
\begin{equation*}
\epsilon_{n l j}^{I}=\left\langle\Phi_{n l j}^{I M}(d)\right| H^{\prime}\left|\Phi_{n l j}^{I M}(d)\right\rangle . \tag{3.8}
\end{equation*}
$$

The off-diagonal matrix elements of $H^{\prime}$ is ignored at this level. Their contribution is however considered when the residual interaction is studied. It is an open interesting question how to determine the mean field operator which admits the energies given by Eq.(3.8) as eigenvalues.

As shown in Ref.[15], the dependence of the new single particle energies on deformation is similar to that shown by the Nilsson model [24]. The quantum numbers in the two schemes are however different. Indeed, here we generate from each j a multiplet of $(2 j+1)$ states distinguished by the quantum number I, which plays the role of the Nilsson quantum number $\Omega$ and runs from $1 / 2$ to j and, moreover, the energies corresponding to the quantum numbers K and -K are equal to each other.

The deformation dependence of the single particle energies, associated to the projected spherical single particle states, is illustrated in Figs 6, 7 for the protons and neutrons in ${ }^{154} \mathrm{Sm}$, respectively.

On the other hand, for a given I there are $2 I+1$ degenerate sub-states while the Nilsson states are only double degenerate. As explained in Ref.[15], the redundancy problem can be solved by changing the normalization of the model functions:

$$
\begin{equation*}
\left\langle\Phi_{\alpha}^{I M} \mid \Phi_{\alpha}^{I M}\right\rangle=1 \Longrightarrow \sum_{M}\left\langle\Phi_{\alpha}^{I M} \mid \Phi_{\alpha}^{I M}\right\rangle=2 . \tag{3.9}
\end{equation*}
$$

Due to this weighting factor the particle density function is providing the consistency result that the number of particles which can be distributed on the $(2 \mathrm{I}+1)$ sub-states is at most 2 , which agrees


Figure 6: Proton single particle energies are plotted as function of the nuclear deformation parameter $d$ for ${ }^{154} \mathrm{Sm}$.
with the Nilsson model. Here $\alpha$ stands for the set of shell model quantum numbers $n l j$. Due to this normalization, the states $\Phi_{\alpha}^{I M}$ used to calculate the matrix elements of a given operator should be multiplied with the weighting factor $\sqrt{2 /(2 I+1)}$.

Finally, we recall a fundamental result, obtained in Ref.[18], concerning the product of two projected states, which comprises a product of two core components. Therein we have proved that the matrix elements of a two body interaction corresponding to the present scheme are very close to the matrix elements corresponding to spherical states projected from a deformed state consisting of two spherical single particle states times a single collective core wave function. The small discrepancies of the two types of matrix elements could be washed out by using slightly different strengths for the two body interaction in the two methods. This feature is caused by the coherent state properties.

## 4 The model Hamiltonian and the pnQRPA approach

As I have already mentioned in the present lecture we are interested to describe the Gamow-Teller two neutrino double beta decay of an even-even deformed nucleus. The $2 \nu \beta \beta$ process is conceived as two successive single $\beta^{-}$transitions. The first transition connects the ground state of the mother nucleus to a magnetic dipole state $1^{+}$of the intermediate odd-odd nucleus which subsequently decays to the first state $2^{+}$of the daughter nucleus. The second leg of the transition is forbidden within the $p n Q R P A$ approach but non-vanishing within a higher pnQRPA approach [21]. The states, involved in the $2 \nu \beta \beta$ process are described by the following many body Hamiltonian:

$$
H=\quad \sum_{\tau \alpha I M} \frac{2}{2 I+1}\left(\epsilon_{\tau \alpha I}-\lambda_{\tau \alpha}\right) c_{\tau \alpha I M}^{\dagger} c_{\tau \alpha I M}-\sum_{\tau \alpha \alpha^{\prime} I} \frac{G_{\tau}}{4} P_{\tau \alpha I}^{\dagger} P_{\tau \alpha^{\prime} I}
$$



Figure 7: Neutron single particle energies are plotted as function of the nuclear deformation parameter $d$ for ${ }^{154} \mathrm{Sm}$.

$$
\begin{align*}
& +2 \chi \sum_{p n ; p^{\prime} n^{\prime} ; \mu} \beta_{\mu}^{-}(p n) \beta_{-\mu}^{+}\left(p^{\prime} n^{\prime}\right)(-)^{\mu}-2 \chi_{1} \sum_{p n ; p^{\prime} n^{\prime} ; \mu} P_{1 \mu}^{-}(p n) P_{1,-\mu}^{+}\left(p^{\prime} n^{\prime}\right)(-)^{\mu} \\
& -\sum_{\tau, \tau^{\prime}=p, n} X_{\tau, \tau^{\prime}} Q_{\tau} Q_{\tau^{\prime}}^{\dagger} . \tag{4.1}
\end{align*}
$$

The operator $c_{\tau \alpha I M}^{\dagger}\left(c_{\tau \alpha I M}\right)$ creates (annihilates) a particle of type $\tau(=\mathrm{p}, \mathrm{n})$ in the state $\Phi_{\alpha}^{I M}$, when acting on the vacuum state $|0\rangle$. In order to simplify the notations, hereafter the set of quantum numbers $\alpha(=n l j)$ will be omitted. The two body interaction consists of three terms, the pairing, the dipole-dipole particle-hole ( ph ) and the particle-particle ( pp ) interactions. The corresponding strengths are denoted by $G_{\tau}, \chi, \chi_{1}$, respectively. All of them are separable interactions, with the factors defined by the following expressions:

$$
\begin{align*}
P_{\tau I}^{\dagger} & =\sum_{M} \frac{2}{2 I+1} c_{\tau I M}^{\dagger} c_{\tau I M}^{\dagger}, \\
\beta_{\mu}^{-}(p n) & =\sum_{M, M^{\prime}} \frac{\sqrt{2}}{\hat{I}}\langle p I M| \sigma_{\mu}\left|n I^{\prime} M^{\prime}\right\rangle \frac{\sqrt{2}}{\hat{I}^{\prime}} c_{p I M^{\prime}}^{\dagger} c_{n I^{\prime} M^{\prime}}, \\
P_{1 \mu}^{-}(p n) & =\sum_{M, M^{\prime}} \frac{\sqrt{2}}{\hat{I}}\langle p I M| \sigma_{\mu}\left|n I^{\prime} M^{\prime}\right\rangle \frac{\sqrt{2}}{\hat{I}^{\prime}} c_{p I M}^{\dagger} c_{n I^{\prime} M^{\prime}}^{\dagger} \\
Q_{2 \mu}^{(\tau)} & =\sum_{i, k} q_{i k}^{(\tau)}\left(c_{i}^{\dagger} c_{k}\right)_{2 \mu}, q_{i k}^{(\tau)}=\frac{\sqrt{2}}{\hat{I}_{k}}\left\langle I_{i} \| r^{2} Y_{2}\right|\left|I_{k}\right\rangle . \tag{4.2}
\end{align*}
$$

The remaining operators from Eq.(4.1) can be obtained from the above operators, by hermitian conjugation.

The one body term and the pairing interaction terms are treated first through the standard BCS formalism and consequently replaced by the quasiparticle one body term $\sum_{\tau I M} E_{\tau} a_{\tau I M}^{\dagger} a_{\tau I M}$. In terms of quasiparticle creation $\left(a_{\tau I M}^{\dagger}\right)$ and annihilation $\left(a_{\tau I M}\right)$ operators, related to the particle operators by means of the Bogoliubov-Valatin transformation, the two body interaction terms, involved in the model Hamiltonian, can be expressed just by replacing the operators (3.2) by their quasiparticle images. Thus, the Hamiltonian terms describing the quasiparticle correlations become a quadratic expression in the dipole and quadrupole two quasiparticles and quasiparticle density operators:

$$
\begin{align*}
A_{1 \mu}^{\dagger}(p n) & =\sum_{m_{p}, m_{n}} C_{m_{p} m_{n} \mu}^{I_{p} I_{n} 1} a_{p I_{p} m_{p}}^{\dagger} a_{n I_{n} m_{n}}^{\dagger}, \\
B_{1 \mu}^{\dagger}(p n) & =\sum_{m_{p}, m_{n}} C_{m_{p}-m_{n}}^{I_{p} I_{n} 1} \mu a_{p I_{p} m_{p}}^{\dagger} a_{n I_{n} m_{n}}(-)^{I_{n}-m_{n}}, \\
A_{2 \mu}^{\dagger}\left(\tau \tau^{\prime}\right) & =\sum_{m_{\tau}, m_{\tau^{\prime}}} C_{m_{\tau} m_{\tau^{\prime}}}^{I_{\tau} I_{I^{\prime}} 2} \mu a_{\tau I_{\tau} m_{\tau}}^{\dagger} a_{\tau^{\prime} I_{\tau^{\prime}} m_{\tau^{\prime}}}^{\dagger},  \tag{4.3}\\
B_{2 \mu}^{\dagger}\left(\tau \tau^{\prime}\right) & =\sum_{m_{\tau}, m_{\tau^{\prime}}} C_{m_{\tau}-m_{\tau^{\prime}} \mu}^{I_{\tau} I_{I^{\prime}} 2} a_{\tau I_{\tau} m_{\tau}}^{\dagger} a_{\tau^{\prime} I_{\tau^{\prime}} m_{\tau^{\prime}}}(-)^{I_{\tau^{\prime}}-m_{\tau^{\prime}}}, \tau, \tau^{\prime}=p, n
\end{align*}
$$

The basic operators defining the model Hamiltonian can be expressed as:

$$
\begin{align*}
\beta_{\mu}^{-}(k) & =\sigma_{k} A_{1 \mu}^{\dagger}(k)+\bar{\sigma}_{k} A_{1,-\mu}(k)(-)^{1-\mu}+\eta_{k} B_{1 \mu}^{\dagger}(k)-\bar{\sigma}_{k} B_{1,-\mu}(k)(-)^{1-\mu}, \\
\beta_{\mu}^{+}(k) & =-\left[\bar{\sigma}_{k} A_{1 \mu}^{\dagger}(k)+\sigma_{k} A_{1,-\mu}(k)(-)^{1-\mu}-\bar{\eta}_{k} B_{1 \mu}^{\dagger}(k)+\sigma_{k} B_{1,-\mu}(k)(-)^{1-\mu}\right], \\
P_{1 \mu}^{-}(k) & =\eta_{k} A_{1 \mu}^{\dagger}(k)-\bar{\eta}_{k} A_{1,-\mu}(k)(-)^{1-\mu}-\sigma_{k} B_{1 \mu}^{\dagger}(k)+\bar{\sigma}_{k} B_{1,-\mu}(k)(-)^{1-\mu}, \\
P_{\mu}^{+}(k) & =-\left[-\bar{\eta}_{k} A_{1 \mu}^{\dagger}(k)+\eta_{k} A_{1,-\mu}(k)(-)^{1-\mu}+\bar{\sigma}_{k} B_{1 \mu}^{\dagger}(k)-\sigma_{k} B_{1,-\mu}(k)(-)^{1-\mu}\right] . \tag{4.4}
\end{align*}
$$

In the above equations the argument "k" stands for the proton-neutron state ( $\mathrm{p}, \mathrm{n}$ ). Here, the usual notations for the dipole two quasiparticle and quasiparticle density operator have been used:

The coefficients $\sigma$ and $\eta$ are simple expressions of the reduced matrix elements of the Pauli matrix $\sigma$ and U and V coefficients:

$$
\begin{gather*}
\sigma_{k}=\frac{2}{\hat{1} \hat{I}_{n}}\left\langle I_{p}\|\sigma\| I_{n}\right\rangle U_{I_{p}} V_{I_{n}}, \quad \bar{\sigma}_{k}=\frac{2}{\hat{1} \hat{I}_{n}}\left\langle I_{p}\|\sigma\| I_{n}\right\rangle V_{I_{p}} U_{I_{n}}, \\
\eta_{k}=\frac{2}{\hat{1} \hat{I}_{n}}\left\langle I_{p}\|\sigma\| I_{n}\right\rangle U_{I_{p}} U_{I_{n}}, \quad \bar{\eta}_{k}=\frac{2}{\hat{1} \hat{I}_{n}}\left\langle I_{p}\|\sigma\| I_{n}\right\rangle V_{I_{p}} V_{I_{n}}, \tag{4.5}
\end{gather*}
$$

The model Hamiltonian, written in terms of quasiparticle operators, is further treated by the pnQRPA formalism

### 4.1 The pnQRPA formalism

The dipole proton-neutron phonon operator has the expression [19, 20]:

$$
\begin{equation*}
\Gamma_{1 \mu}^{\dagger}=\sum_{k}\left[X_{1}(k) A_{1 \mu}^{\dagger}(k)-Y_{1}(k) A_{1,-\mu}(k)(-)^{1-\mu}\right] . \tag{4.6}
\end{equation*}
$$

and satisfies the restrictions:

$$
\begin{equation*}
\left[\Gamma_{1 \mu}, \Gamma_{1 \mu^{\prime}}^{\dagger}\right]=\delta_{\mu, \mu^{\prime}}, \quad\left[H_{q p}, \Gamma_{1 \mu}^{\dagger}\right]=\omega \Gamma_{1 \mu}^{\dagger} . \tag{4.7}
\end{equation*}
$$

These operator equations yield a set of algebraic equations for the X (usually called forward going) and Y (named back-going) amplitudes:

$$
\begin{gather*}
\left(\begin{array}{cc}
\mathcal{A} & \mathcal{B} \\
-\mathcal{B} & -\mathcal{A}
\end{array}\right)\binom{X}{Y}=\omega\binom{X}{Y}  \tag{4.8}\\
\sum_{k}\left[|X(k)|^{2}-|Y(k)|^{2}\right]=1 \tag{4.9}
\end{gather*}
$$

The pnQRPA matrices $\mathcal{A}$ and $\mathcal{B}$ have analytical expressions:

$$
\begin{align*}
\mathcal{A}_{k, k^{\prime}} & =\left(E_{p}+E_{n}\right) \delta_{p p^{\prime}} \delta_{n n^{\prime}}+2 \chi\left(\sigma_{k} \sigma_{k^{\prime}}+\bar{\sigma}_{k} \bar{\sigma}_{k^{\prime}}\right)-2 \chi_{1}\left(\eta_{k} \eta_{k^{\prime}}+\bar{\eta}_{k} \bar{\eta}_{k^{\prime}}\right) \\
\mathcal{B}_{k, k^{\prime}} & =2 \chi\left(\bar{\sigma}_{k} \sigma_{k^{\prime}}+\sigma_{k} \bar{\sigma}_{k^{\prime}}\right)+2 \chi_{1}\left(\bar{\eta}_{k} \eta_{k^{\prime}}+\eta_{k} \bar{\eta}_{k^{\prime}}\right) \tag{4.10}
\end{align*}
$$

All quantities involved in the pnQRPA matrices have been already defined. Note that the proton and neutron quasiparticle energies are denoted in an abbreviated manner by $E_{p}$ and $E_{n}$, respectively.

The charge conserving $Q R P A$ bosons

$$
\begin{equation*}
\Gamma_{2 \mu}^{\dagger}=\sum_{k}\left[X_{2}(k) A_{2 \mu}^{\dagger}(k)-Y_{1}(k) A_{2,-\mu}(k)(-)^{\mu}\right], k=\left(p, p^{\prime}\right),\left(n, n^{\prime}\right) \tag{4.11}
\end{equation*}
$$

are determined by the $Q R P A$ equations associated to the matrices:

$$
\begin{align*}
\mathcal{A}_{\tau \tau^{\prime}}\left(i k ; i^{\prime} k^{\prime}\right) & =\delta_{\tau \tau^{\prime}} \delta_{i i^{\prime}} \delta_{k k^{\prime}}\left(E_{i}^{\tau}+E_{k}^{\tau}\right)-X_{\tau \tau^{\prime}}\left(q_{i k}^{(\tau)} \xi_{i k}^{(\tau)}\right)\left(q_{i^{\prime} k^{\prime}}^{(\tau)} \xi_{i^{\prime} k^{\prime}}^{(\tau)}\right) \\
\mathcal{B}_{\tau \tau^{\prime}}\left(i k ; i^{\prime} k^{\prime}\right) & =-X_{\tau \tau^{\prime}}\left(q_{i k}^{(\tau)} \xi_{i k}^{(\tau)}\right)\left(q_{i^{\prime} k^{\prime}}^{(\tau)} \xi_{i^{\prime} k^{\prime}}^{(\tau)}\right), i \leq k, i^{\prime} \leq k^{\prime} \tag{4.12}
\end{align*}
$$

where

$$
\begin{equation*}
\xi_{i k}^{(\tau)}=\frac{1}{\sqrt{1+\delta_{i, k}}}\left(U_{i}^{\tau} V_{k}^{\tau}+U_{k}^{\tau} V_{i}^{\tau}\right) \tag{4.13}
\end{equation*}
$$

Here $V_{i}^{\tau}$ and $U_{i}^{\tau}$ denote the square roots of occupation and non-occupation probabilities of the state $i$ of $\tau(=\mathrm{p}, \mathrm{n})$ type respectively, given by the BCS equations. In order to distinguish between the phonon operators acting in the RPA space associated to the mother and daughter nuclei respectively, one needs an additional index. Also, an index labeling the solutions of the RPA equations is necessary. Thus, the two kinds of bosons will be denoted by:

$$
\begin{equation*}
{ }_{j} \Gamma_{1 \mu}^{\dagger}(k), j=i, f ; k=1,2, \ldots N_{s}^{(1)} ;{ }_{j} \Gamma_{2 \mu}^{\dagger}(k), j=i, f ; k=1,2, \ldots N_{s}^{(2)} . \tag{4.14}
\end{equation*}
$$

Acting with ${ }_{i} \Gamma_{1 \mu}^{\dagger}(k)$ and ${ }_{f} \Gamma_{1 \mu}^{\dagger}(k)$ on the vacuum states $|0\rangle_{i}$ and $|0\rangle_{f}$ respectively, one obtains two sets of non-orthogonal states describing the intermediate odd-odd nucleus. By contrast, the states ${ }_{i} \Gamma_{2}^{\dagger}(k)|0\rangle_{i}$ and ${ }_{f} \Gamma_{2}^{\dagger}(k)|0\rangle_{f}$ describe different nuclei, namely the initial and final ones, participating in the process of $2 \nu \beta \beta$ decay. The mentioned indices are however omitted whenever their presence is not necessary.

### 4.2 Going beyond pnQRPA; The Boson Expansion (BE) procedure

Within the boson expansion formalism, the basic operators $A_{1 \mu}^{\dagger}(p, n), A_{1 \mu}, B_{1 \mu}^{\dagger}(p, n), B_{1 \mu}$ are written as polynomial expansions in terms of the QRPA boson operators with the expansion coefficients determined such that their mutual commutation relations are preserved in each order of approximation [25]. Based on this criterion the boson expansions of the quadrupole two quasiparticle and quadrupole quasiparticle density charge conserving operators have been obtained by Belyaev and Zelevinsky in Ref.[25]. For charge non-conserving two quasiparticle and quasiparticle density dipole operators the expansion has been derived by one of us (A.A.R, in collaboration) in Ref.([21]). The latter expansion has the peculiarity that the commutator algebra cannot be satisfied restricting the expansion to the proton-neutron dipole bosons. However, this goal can be touched if the boson operators space is enlarged by adding the charge conserving quadrupole two quasiparticle bosons. The last step consists
in expressing the quasi-boson operators $\stackrel{0}{A}_{1 \mu}^{\dagger}(p n), \stackrel{0}{A_{1 \mu}}(p n), \stackrel{0^{\dagger}}{A_{2 \mu}}(p p), \stackrel{0^{\dagger}}{A_{2 \mu}}(p p),{\stackrel{0}{A} A_{2 \mu}^{\dagger}}_{(n n),{ }_{A}^{A_{2 \mu}}(n n)}$ (these are, in fact the operators denoted by the same symbol but without the index "0", with the commutators approximated to be of boson type) as linear combination of the QRPA bosons. In this way the basic operators mentioned above are written as polynomials of $p n$ and $p p+n n$ QRPA bosons. The expansions involve not only the collective but also non-collective QRPA bosons. The final expressions obtained in this way are:

$$
\begin{align*}
& A_{1 \mu}^{\dagger}\left(j_{p} j_{n}\right)= \\
+ & \sum_{k_{1}}\left\{\mathcal{A}_{k_{1}}^{(1,0)}\left(j_{p} j_{n}\right) \Gamma_{1 \mu}^{\dagger}\left(k_{1}\right)+\mathcal{A}_{k_{1}}^{(0,1)}\left(j_{p} j_{n}\right) \Gamma_{1-\mu}\left(k_{1}\right)(-)^{1-\mu}\right\} \\
+ & \sum_{k_{1}, k_{2}, k_{3} ; l=0,2}\left\{\mathcal{A}_{k_{3} k_{2} k_{1}}^{(3,0) l}\left(j_{p} j_{n}\right)\left[\left(\Gamma_{2}^{\dagger}\left(k_{3}\right) \Gamma_{2}^{\dagger}\left(k_{2}\right)\right)_{l} \Gamma_{1}^{\dagger}\left(k_{1}\right)\right]_{1 \mu}+\mathcal{A}_{k_{3} k_{2} k_{1}}^{(0,3) ; l}\left(j_{p} j_{n}\right)\left[\left(\Gamma_{2}\left(k_{3}\right) \Gamma_{2}\left(k_{2}\right)\right)_{l} \Gamma_{1}\left(k_{1}\right)\right]_{1 \mu}\right\} \\
& \left\{\mathcal{A}_{k_{1}, k_{2}, k_{3} ; l=0,2}^{1 ;(2 \overline{2}) l}\left(j_{p} j_{n}\right)\left[\Gamma_{1}^{\dagger}\left(k_{1}\right)\left(\Gamma_{2}^{\dagger}\left(k_{2}\right) \Gamma_{2}\left(k_{3}\right)\right)_{l}\right]_{1 \mu}+\mathcal{A}_{k_{3} k_{2} k_{1}}^{(2 \overline{2}) l i}\left(j_{p} j_{n}\right)\left[\left(\Gamma_{2}^{\dagger}\left(k_{3}\right) \Gamma_{2}\left(k_{2}\right)\right)_{l} \Gamma_{1}\left(k_{1}\right)\right]_{1 \mu}\right\} \\
& B_{1 \mu}^{\dagger}\left(j_{p} j_{n}\right)=\sum_{k_{1} k_{2}}\left\{\mathcal{B}_{k_{1} k_{2}}^{(2,0)}\left(j_{p} j_{n}\right)\left[\Gamma_{1}^{\dagger}\left(k_{1}\right) \Gamma_{2}^{\dagger}\left(k_{2}\right)\right]_{l \mu}+\mathcal{B}_{k_{1} k_{2}}^{(0,2)}\left(j_{p} j_{n}\right)\left[\Gamma_{1}\left(k_{1}\right) \Gamma_{2}\left(k_{2}\right)\right]_{l \mu}\right.  \tag{4.15}\\
+ & \left.\mathcal{B}_{k_{1} k_{2}}^{11 ; 12}\left(j_{p} j_{n}\right)\left[\Gamma_{1}^{\dagger}\left(k_{1}\right) \Gamma_{2}\left(k_{2}\right)\right]_{l \mu}+\mathcal{B}_{k_{1} k_{2}}^{11 ; 2 l}\left(j_{p} j_{n}\right)\left[\Gamma_{1}^{\dagger}\left(k_{2}\right) \Gamma_{1}\left(k_{1}\right)\right]_{l \mu}\right\},
\end{align*}
$$

where the expansion coefficients are those given in Ref.[21] while the notations for the dipole and quadrupole bosons introduced in the previous section have been used. The boson expansions associated to the two quasiparticle and quasiparticle density proton-neutron operators have the property that the two sides of Eqs.4.15 have the same matrix elements in a boson basis. Actually, this can be used as a criterion to determine the expansion coefficients. For example the first expansion coefficients in the above expression can be determined as:

$$
\begin{align*}
\mathcal{A}_{k_{1}}^{(1,0)}\left(j_{p} j_{n}\right) & =\langle 0|\left[\Gamma_{1 \mu}\left(k_{1}\right), A_{1 \mu}^{\dagger}\left(j_{p}, j_{n}\right)\right]|0\rangle \\
\mathcal{B}_{k_{1} k_{2}}^{(2,0)}\left(j_{p} j_{n}\right) & =\sum_{\mu_{1}, \mu_{2}} C_{\mu_{1}}^{1}{ }_{\mu}^{2} \mu_{2} \mu \tag{4.16}
\end{align*}\langle 0|\left[\Gamma_{1 \mu_{1}}\left(k_{1}\right),\left[\Gamma_{1 \mu_{2}}\left(k_{2}\right), B_{1 \mu}^{\dagger}\left(j_{p}, j_{n}\right)\right]\right]|0\rangle .
$$

The properties of the nested commutators determine vanishing values for the coefficients accompanying the operators involving an even number of bosons in the $A^{\dagger}$ expansion and an odd number of bosons in the $B^{\dagger}$ expansion. Thus, the $A_{1 \mu}^{\dagger}$ has an odd order boson expansion while $B_{1 \mu}^{\dagger}$ exhibits an even order expansion in bosons. It is worth mentioning that the matrix element of the double commutator involved in Eq.4.16 does not depend on the order in which the commutators are performed. Indeed, the same result is obtained when a) first the $k_{2}$ boson is commuted with $B^{\dagger}$ and the result is commuted with the $k_{1}$ boson and b ) first the dipole boson is commuted with $B^{\dagger}$ and the result is commuted with the quadrupole boson. However, the commutation order is important when one determines the remaining expansion coefficients. The ordering in the mentioned commutators is chosen such that the mutual commutator equations of the basic operators $A_{1 \mu}^{\dagger}, B_{1 \mu}^{\dagger}$ are satisfied in each order of approximation. The comparison of the boson expansion formulated in Ref.[21] and other approaches may be found in Ref.[30]

### 4.3 The GT amplitude for the transition $0^{+} \rightarrow 2^{+}$within the BE formalism

At this stage we can specify the intermediate states involved in the GT amplitude describing the transition $0^{+} \rightarrow 2^{+}$. The intermediate states $|k, m\rangle$ are k -boson states with $k=1,2,3$ labeled by the index m , indicating the spin and the ordering label of the RPA roots. Note that by contrast to the case of ground to ground transition here the denominator has a cubic power which results in obtaining a suppression of the corresponding GT amplitude. Inserting the boson expansions from Eq.(4.15) into the expression of the $\beta^{+}$transition operator one can check that the following non-vanishing factors,
at numerator, show up:

$$
\begin{align*}
& { }_{i}\left\langle 0\| \|_{i} \Gamma_{1}\left(k_{1}\right) \| 1,1_{k_{1}}\right\rangle_{i f}\left\langle 1,1_{k_{2}}\left\|_{f} \Gamma_{1}^{\dagger}\left(k_{2}\right)_{f} \Gamma_{2}(1)\right\| 1,2_{1}\right\rangle_{f} \\
& { }_{i}\left\langle 0\| \|_{i} \Gamma_{1}\left(k_{1}\right)_{i} \Gamma_{2}\left(i k_{2}\right) \| 2,1_{k_{1}} 2_{k_{2}}\right\rangle_{i f}\left\langle 2,1_{j_{1}} 2_{1}\left\|_{f} \Gamma_{1}^{\dagger}\left(j_{1}\right)\right\| 1,2_{1}\right\rangle_{f} \\
& { }_{i}\left\langle 0\left\|_{i} \Gamma_{1}\left(k_{1}\right)_{i} \Gamma_{2}\left(k_{2}\right)\right\| 2,1_{k_{1}} 2_{k_{2}}\right\rangle_{i f}\left\langle 2,1_{j_{1}} 2_{j_{2}}\| \|_{f} \Gamma_{1}^{\dagger}\left(j_{1}\right)_{f} \Gamma_{2}^{\dagger}\left(j_{2}\right)_{f} \Gamma_{2}(1) \| 1,2_{1}\right\rangle_{f} \\
& { }_{i}\left\langle 0\| \|_{i} \Gamma_{1}\left(k_{1}\right)_{i} \Gamma_{2}\left(k_{2}\right)_{i} \Gamma_{2}\left(k_{3}\right) \| 3,1_{k_{1}} 2_{k_{2}} 2_{k_{3}}\right\rangle_{i f}\left\langle 3,1_{j_{1}} 2_{j_{2}} 2_{1}\left\|_{f} \Gamma_{1}^{\dagger}\left(j_{1}\right)_{f} \Gamma_{2}^{\dagger}\left(j_{2}\right)\right\| 1,2_{1}\right\rangle_{f} . \tag{4.17}
\end{align*}
$$

The term $E_{k, m}$ from the denominator of Eq. (2.6) is the average of the energies of the mother and daughter states $|k, m\rangle$ normalized to the average energy of the first $p n Q R P A$ states $1^{+}$in the initial and final nuclei. The left low indices " $i$ " and " $f$ " suggest that the phonon operators are built up with quasiparticle operators characterizing the initial and final nuclei, respectively. Acting with the $i$ and $f$ dipole single or dipole multi-phonon operators on the states $|0\rangle_{i}$ and $\left|2_{f}^{+}\right\rangle\left(\right.$or $\left.|0\rangle_{f}\right)$ one populates two sets of states $\left|1^{+}\right\rangle_{i}$ and $\left|1^{+}\right\rangle_{f}$ respectively, characterizing the odd-odd intermediate nuclei. The two sets are not orthogonal onto each other.

The matrix elements, listed above, are associated to partial transition amplitudes represented pictorially in Fig.8.

## 5 Numerical application

The formalism described in the previous section, has been applied to eighteen isotopes which have been earlier considered in Refs.[19, 20] for studying the double beta ground to ground transition. Among these, eleven are proved to be, indeed, double beta ground to ground emitters, while the remaining ones are suspected to have this property due to the corresponding positive Q-value. Since the excitation energies for the states $2^{+}$in the daughter nuclei are not large, the Q-values characterizing the double beta transition $0^{+} \rightarrow 2^{+}$are also positive. For some of the selected nuclei, experimental data either for the half life of the process or for the low bounds of the half lives are available.

In order to save the space, here we shall present the results of our calculations for ten double beta emitters: ${ }^{48} \mathrm{Ca},{ }^{96} \mathrm{Zr},{ }^{100} \mathrm{Mo},{ }^{104} \mathrm{Ru},{ }^{110} \mathrm{Pd},{ }^{116} \mathrm{Cd},{ }^{128} \mathrm{Te},{ }^{130} \mathrm{Te},{ }^{134} \mathrm{Xe},{ }^{136} \mathrm{Xe}$.

### 5.1 Fixing the parameters involved by the model Hamiltonian

### 5.1.1 The mean field parameters

The spherical shell model parameters for these double beta emitters and the corresponding daughter nuclei are given by:

$$
\begin{equation*}
\hbar \omega_{0}=41 A^{1 / 3}, \quad C=2 \hbar \omega_{0} \kappa, \quad D=\hbar \omega_{0} \mu \tag{5.1}
\end{equation*}
$$

with the strength parameters $\kappa$ and $\mu$ having the same (Z,N) dependence as in Ref. [32].
The angular momentum projected basis depends on two additional parameters. These are the deformation parameter $d$ and the factor $k$ entering the canonical transformation relating the quadrupole coordinate and boson operators $[19,20]$. They were fixed in the following manner: We require that the relative energy for the states $\left|1 f \frac{7}{2} \frac{7}{2}\right\rangle$ and $\left|1 d \frac{5}{2} \frac{1}{2}\right\rangle$ be equal to that of Nilsson levels with $\Omega=\frac{7}{2}$ and $\Omega=\frac{1}{2}$ in the $N=3$ major shell. Moreover, adding to the mean field term defined before a $Q Q$ two body interaction we require that the lowest root for the charge conserving QRPA equation be equal to the experimental energy of the lowest $2^{+}$state in the mother nucleus. Throughout this paper, the M-degenerate states $\Phi_{n l j}^{I M}$ are denoted by $|n+1 l j I\rangle$.

### 5.1.2 The strength of pairing interactions

The BCS calculation has been performed within a restricted single particle space. Due to the level crossing, the restriction of the single particle space for deformed nuclei is different from that for spherical nuclei. Indeed, in spherical nuclei Ikeda sum rule (ISR) is satisfied if two major shells plus the spin orbit partner of the intruder state are included in the single particle space. Suppose that


Figure 8: One illustrates various GT transitions $0^{+} \rightarrow 2^{+}$via one (a)) two (b)) and (c)) and three (d)) phonon states.
the neutron open shell has $\mathrm{N}=3$ with the intruder state $|1 g 9 / 2\rangle$, in the standard spherical shell model picture. In the present formalism, including the spin-orbit partner state $|1 g 7 / 2\rangle$ means to consider the states $\Phi_{0,4, \frac{7}{2}}^{I M}$ with $I=7 / 2,5 / 2,3 / 2,1 / 2$. However, some of these states are higher in energy than states belonging to the $|2 d 5 / 2 I\rangle$ multiplet. Due to such features appearing both in the upper part of the major open shell of neutrons and the bottom side of the proton major open shell we truncated the space considering an inert ( $\mathrm{Z}, \mathrm{N}$ ) core and a number of states lying above the core states. The core and the number of outside states are chosen such that the non-occupation probabilities for the neglected bottom states as well as the occupation probabilities for the ignored upper states are smaller than 0.01 . Of course, the single particle space for protons and neutrons are the same. Our calculations were performed with the core and number of states given in Table I. Once the single particle space is defined, the number of the dipole proton-neutron states can be calculated. Furthermore, the dimensions of the pnQRPA matrices for mother $\left(D_{1}\right)$ and daughter $\left(D_{2}\right)$ nuclei are readily obtained. These dimensions are also given in Table I. It is worth mentioning that using the single particle spaces given in Table I,

Ikeda sum rule is satisfied for both the mother and daughter nuclei considered in the present paper.

| Nucleus | ${ }^{48} \mathrm{Ca}$ | ${ }^{96} \mathrm{Zr}$ | ${ }^{100} \mathrm{Mo}$ | ${ }^{104} \mathrm{Ru}$ | ${ }^{110} \mathrm{Pd}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| The (Z,N) core | $(0,0)$ | $(20,20)$ | $(26,26)$ | $(26,26)$ | $(26,26)$ |
| Number of states | 19 | 20 | 20 | 22 | 23 |
| $D_{1}$ | 118 | 128 | 132 | 140 | 154 |
| $D_{2}$ | 115 | 128 | 132 | 140 | 154 |


| Nucleus | ${ }^{116} \mathrm{Cd}$ | ${ }^{128} \mathrm{Te}$ | ${ }^{130} \mathrm{Te}$ | ${ }^{134} \mathrm{Xe}$ | ${ }^{136} \mathrm{Xe}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| The (Z,N) core | $(26,26)$ | $(44,44)$ | $(44,44)$ | $(44,44)$ | $(44,44)$ |
| Number of states | 27 | 22 | 23 | 21 | 21 |
| $D_{1}$ | 166 | 142 | 150 | 138 | 154 |
| $D_{2}$ | 166 | 128 | 132 | 120 | 140 |

Table 1: The number of single particle proton states lying above the (Z,N) core is given. The single particle space for neutrons is identical to that for protons. $D_{1}$ and $D_{2}$ are the dimensions of the pnQRPA matrix for mother and daughter nuclei, respectively.

The dimensions of the single particle basis used in our calculations are presented in Table 1. Note that an inert core is assumed for each case and a limited number of states above the considered core were taken into account.

The paring strengths have been fixed so that the gap energies known in the mother and daughter nuclei are reproduced. The results are given in Table II.

| Nucleus | d | k | $\mathrm{G}_{\mathrm{p}}[\mathrm{MeV}]$ | $\mathrm{G}_{\mathrm{n}}[\mathrm{MeV}]$ | $\chi[\mathrm{MeV}]$ | $\mathrm{g}_{\mathrm{pp}}$ | $\left(\frac{1}{2} Q_{\beta \beta}+m_{e} c^{2}\right)[\mathrm{MeV}]$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| ${ }^{48} \mathrm{Ca}$ | 0.3 | 10.00 | 0.65 | 0.45 | 0.180 | 0.0 | 2.646 |
| ${ }^{48} \mathrm{Ti}$ | 0.05 | 2.00 | 0.46 | 0.36 | 0.180 | 0.0 |  |
| ${ }^{96} \mathrm{Zr}$ | 1.5 | 10.20 | 0.26 | 0.26 | 0.5 | 0.112 | 2.186 |
| ${ }^{96} \mathrm{Mo}$ | 1.2 | 7.20 | 0.3 | 0.3 | 0.5 | 0.112 |  |
| ${ }^{100} \mathrm{Mo}$ | -1.4 | 10.00 | 0.28 | 0.26 | 0.050 | 0.112 | 2.026 |
| ${ }^{100} \mathrm{Ru}$ | -0.6 | 3.6 | 0.285 | 0.220 | 0.050 | 0.112 |  |
| ${ }^{104} \mathrm{Ru}$ | -1.55 | 8.80 | 0.26 | 0.2 | 0.150 | 2.750 | 1.161 |
| ${ }^{104} \mathrm{Pd}$ | -1.35 | 6.94 | 0.26 | 0.180 | 0.150 | 2.750 |  |
| ${ }^{110} \mathrm{Pd}$ | -1.6 | 6.00 | 0.30 | 0.32 | 0.148 | 2.450 | 1.516 |
| ${ }^{110} \mathrm{Cd}$ | -0.8 | 3.06 | 0.30 | 0.18 | 0.148 | 2.450 |  |
| ${ }^{116} \mathrm{Cd}$ | -1.8 | 3.00 | 0.20 | 0.245 | 0.187 | 2.120 | 1.916 |
| ${ }^{116} \mathrm{Sn}$ | -1.2 | 2.50 | 0.18 | 0.275 | 0.187 | 2.120 |  |
| ${ }^{128} \mathrm{Te}$ | 0.5 | 1.62 | 0.27 | 0.22 | 0.265 | 1.908 | 0.946 |
| ${ }^{128} \mathrm{Xe}$ | 1.7 | 6.50 | 0.23 | 0.22 | 0.265 | 1.908 |  |
| ${ }^{130} \mathrm{Te}$ | 0.493 | 1.88 | 0.24 | 0.21 | 0.280 | 1.895 | 1.776 |
| ${ }^{130} \mathrm{Xe}$ | 1.4 | 5.00 | 0.24 | 0.205 | 0.280 | 1.895 |  |
| ${ }^{134} \mathrm{Xe}$ | -0.1 | 1.95 | 0.28 | 0.30 | 0.101 | 0.0 | 0.931 |
| ${ }^{134} \mathrm{Ba}$ | -0.468 | 1.50 | 0.24 | 0.24 | 0.101 | 0.0 |  |
| ${ }^{136} \mathrm{Xe}$ | -0.1 | 1.80 | 0.23 | 0.29 | 0.190 | 2.25 | 1.751 |
| ${ }^{136} \mathrm{Ba}$ | -0.698 | 2.16 | 0.19 | 0.20 | 0.190 | 2.25 |  |

Table 2: The pairing and Gamow Teller $p h$ interaction strengths are given in units of MeV . The ratio of the two dipole interaction ( particle-hole and particle-particle) strengths, denoted by $g_{p p}$, is also given.

### 5.1.3 The long range Q.Q interaction

The microscopic Hamiltonian used for describing the double beta $0^{+} \rightarrow 2^{+}$transition, involves in addition to the terms considered in the treatment of the ground to ground transition, the quadrupolequadrupole interaction between alike nucleons. As we already mentioned this interaction is needed in order to define the charge conserving quadrupole phonon operators used by the boson expansion procedure. Moreover, this interaction is used to describe the final state, i.e. $2^{+}$, in the daughter nucleus. The strength of the $Q Q$ interaction was fixed by requiring that the first root of the QRPA equation for the quadrupole charge conserving boson is close to the experimental energy of the first $2^{+}$state. The results of the fitting procedure are given in Table III.

| Nucleus | $E_{2+}^{\text {exp. }}[\mathrm{keV}]$ | $E_{2^{+}}^{\text {th. }}[\mathrm{keV}]$ | $b^{4} X_{p p}[\mathrm{keV}]$ | Nucleus | $E_{2+}^{\text {exp. }}[\mathrm{keV}]$ | $E_{2^{+}}^{\text {th. }}[\mathrm{keV}]$ | $b^{4} X_{p p}[\mathrm{keV}]$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| ${ }^{48} \mathrm{Ca}$ | 983.00 | 983.00 | 71.30 | ${ }^{130} \mathrm{Te}$ | 839.49 | 831.03 | 12.12 |
| ${ }^{48} \mathrm{Ti}$ | 983.52 | 979.02 | 42.80 | ${ }^{130} \mathrm{Xe}$ | 536.07 | 534.2 | 17.28 |
| ${ }^{76} \mathrm{Ge}$ | 562.93 | 558.88 | 50.80 | ${ }^{134} \mathrm{Xe}$ | 847.04 | 841.75 | 20.00 |
| ${ }^{76} \mathrm{Se}$ | 559.10 | 558.87 | 65.20 | ${ }^{134} \mathrm{Ba}$ | 604.72 | 607.98 | 17.56 |
| ${ }^{82} \mathrm{Se}$ | 654.75 | 654.73 | 19.10 | ${ }^{136} \mathrm{Xe}$ | 1313.027 | 1314.90 | 16.37 |
| ${ }^{82} \mathrm{Kr}$ | 776.52 | 776.84 | 25.84 | ${ }^{136} \mathrm{Ba}$ | 818.49 | 810.30 | 14.82 |
| ${ }^{96} \mathrm{Zr}$ | 1750.49 | 1465.62 | 2.00 | ${ }^{148} \mathrm{Nd}$ | 301.702 | 298.00 | 24.29 |
| ${ }^{96} \mathrm{Mo}$ | 778.24 | 776.81 | 38.10 | ${ }^{148} \mathrm{Sm}$ | 550.250 | 553.00 | 24.64 |
| ${ }^{100} \mathrm{Mo}$ | 535.57 | 534.43 | 31.50 | ${ }^{150} \mathrm{Nd}$ | 130.21 | 135.24 | 27.32 |
| ${ }^{100} \mathrm{Ru}$ | 539.5 | 536.11 | 19.70 | ${ }^{150} \mathrm{Sm}$ | 330.86 | 333.12 | 22.45 |
| ${ }^{104} \mathrm{Ru}$ | 358.03 | 358.45 | 29.80 | ${ }^{154} \mathrm{Sm}$ | 81.976 | 83.05 | 22.62 |
| ${ }^{104} \mathrm{Pd}$ | 555.81 | 561.83 | 20.90 | ${ }^{154} \mathrm{Gd}$ | 123.070 | 123.35 | 20.37 |
| ${ }^{110} \mathrm{Pd}$ | 373.8 | 370.45 | 44.65 | ${ }^{160} \mathrm{Gd}$ | 75.26 | 73.13 | 18.70 |
| ${ }^{110} \mathrm{Cd}$ | 657.76 | 662.85 | 25.10 | ${ }^{160} \mathrm{Dy}$ | 86.788 | 87.25 | 19.03 |
| ${ }^{116} \mathrm{Cd}$ | 513.49 | 514.50 | 30.50 | ${ }^{232} \mathrm{Th}$ | 49.369 | 48.32 | 15.25 |
| ${ }^{116} \mathrm{Sn}$ | 1293.56 | 1179.16 | 7.00 | ${ }^{232} \mathrm{U}$ | 47.572 | 45.22 | 14.94 |
| ${ }^{128} \mathrm{Te}$ | 743.22 | 746.12 | 12.12 | ${ }^{238} \mathrm{U}$ | 44.916 | 47.34 | 12.91 |
| ${ }^{128} \mathrm{Xe}$ | 442.91 | 449.58 | 19.43 | ${ }^{238} \mathrm{Pu}$ | 44.076 | 46.15 | 14.83 |

Table 3: The experimental and calculated energies for the first $2^{+}$states in mother and daughter nuclei are given. The strength parameter of the quadrupole-quadrupole interaction was fixed such that the experimental energies are reproduced. In our calculations we considered $X_{p p}=X_{n n}=X_{p n}$. The oscillator length is denoted by $b=(\hbar / M \omega)^{1 / 2}$.

### 5.1.4 The $p h$ and $p p$ dipole interaction strengths

The $p p$ and $p h$ interaction are related by the so called Pandya transformation. This is the reason the two interactions have not been considered as independent interactions. Consequently the pnQRPA approach took into account only the $p h$ interaction. This happened until 1984 when Cha[33] noticed that the matrix element describing the single $\beta^{+}$transition is very sensitive to the variation of the $p p$ interaction strength. This reference determined people working on the double beta decay issue to use the $p h$ and $p p$ interactions as independent interaction. One expects therefore that the $p p$ interaction influences the second leg of the double beta decay. On the other hand the centroid of the Gamow-Teller giant resonance is very sensitive to the strength of the $p h$ interaction but practically is insensitive to the variation of the $p p$ interaction.

The strengths of the dipole proton-neutron interaction might be taken as in Ref.[34] although the single particle basis used therein, is different from ours:

$$
\begin{equation*}
\chi=\frac{5.2}{A^{0.7}} \mathrm{MeV}, \quad \chi_{1}=\frac{0.58}{\mathrm{~A}^{0.7}} \mathrm{MeV} . \tag{5.2}
\end{equation*}
$$

The A dependence for the ph interaction strength has been derived by fitting the position of the GT
resonance for ${ }^{40} \mathrm{Ca},{ }^{90} \mathrm{Zr}$ and ${ }^{208} \mathrm{~Pb}$. The strength $\chi_{1}$ has been fixed so that the beta decay half lives of the nuclei with $Z \leq 40$ are reproduced.

A certain caution, however, is necessary when these formulae are used, since the A dependence is conditioned by the mass region [35] as well as by the single particle space [36, 37]. For example, in Ref.[38] the GT resonance centroids in ${ }^{128} \mathrm{Te}$ and ${ }^{130} \mathrm{Te}$, located at 13.7 and 14.1 MeV respectively, are reproduced with the $\chi$ values equal to 0.157 and 0.16 MeV respectively. These values for $\chi$ are different from the predictions of Eq.(5.2) corresponding to $\mathrm{A}=128$ and $\mathrm{A}=130$, respectively. Moreover, as we see from Table II, in the current paper the right position of these GT resonances are obtained, by using $\chi=0.265$ and $\chi=0.280$, respectively.

| Mother nucleus | Transition $\log f t$ | Intermediate nucleus | Transition $\log f t$ | Daughter nucleus |
| :---: | :---: | :---: | :---: | :---: |
| ${ }^{100} \mathrm{Mo}$ <br> Exp. <br> Th. | $\begin{gathered} \frac{\beta^{+} / E C}{\leftarrow} \\ - \\ 4.78 \end{gathered}$ | ${ }^{100} \mathrm{Tc}$ | $\begin{gathered} \stackrel{\beta^{-}}{\rightarrow} \\ 4.66{ }^{a)} \\ 4.62 \end{gathered}$ | ${ }^{100} \mathrm{Ru}$ |
| ${ }^{104} \mathrm{Ru}$ <br> Exp. <br> Th. | $\begin{gathered} \hline \beta^{+} / E C \\ 4.32 b) \\ 4.20 \\ \hline \end{gathered}$ | ${ }^{104} \mathrm{Rh}$ | $\begin{gathered} \stackrel{\beta^{-}}{\rightarrow} \\ 4.55^{b)} \\ 4.62 \end{gathered}$ | ${ }^{104} \mathrm{Pd}$ |
| ${ }^{110} \mathrm{Pd}$ <br> Exp. <br> Th. | $\begin{gathered} \hline \beta^{+} / E C \\ 4.08^{c} \\ 3.86 \\ \hline \end{gathered}$ | ${ }^{110} \mathrm{Ag}$ | $\xrightarrow{\beta^{-}}$ $4.66^{c}$ 4.83 | ${ }^{110} \mathrm{Cd}$ |
| ${ }^{116} \mathrm{Cd}$ <br> Exp. <br> Th. | $\begin{gathered} \beta^{+} / E C \\ \leftarrow \\ - \\ 3.94 \end{gathered}$ | ${ }^{116} \mathrm{In}$ | $\begin{gathered} \xrightarrow[\rightarrow]{\beta^{-}} \\ 4.662 \\ 4.660 \end{gathered}$ | ${ }^{116} \mathrm{Sn}$ |
| ${ }^{128} \mathrm{Te}$ <br> Exp. <br> Th. | $\begin{gathered} \hline \beta^{+} / E C \\ \leftarrow \\ - \\ 4.47 \\ \hline \end{gathered}$ | ${ }^{128} \mathrm{I}$ | $\xrightarrow{\beta^{-}}$ $6.061{ }^{e)}$ 6.063 | ${ }^{128} \mathrm{Xe}$ |
| ${ }^{130} \mathrm{Te}$ <br> Exp. <br> Th. | $\beta^{+} / E C$ - 4.47 | ${ }^{130} \mathrm{I}$ | $\begin{gathered} \stackrel{\beta^{-}}{\rightarrow} \\ - \\ 6.061 \\ \hline \end{gathered}$ | ${ }^{130} \mathrm{Xe}$ |
| ${ }^{134} \mathrm{Xe}$ <br> Exp. <br> Th. | $\begin{gathered} \hline \beta^{+} / E C \\ - \\ \hline .38 \\ \hline \end{gathered}$ | ${ }^{134} \mathrm{Cs}$ | $\begin{gathered} \underset{\rightarrow}{\beta^{-}} \\ - \\ 6.07 \\ \hline \end{gathered}$ | ${ }^{134} \mathrm{Ba}$ |

Table 4: The experimental and theoretical $\log f t$ values characterizing the $\beta^{+} / E C$ and $\beta^{-}$processes of the intermediate nucleus ground state $\left(1^{+}\right)$. Experimental data are from: ${ }^{a)}[39],{ }^{b)}[40],{ }^{c}{ }^{c}[41],{ }^{d)}[42]$, ${ }^{e)}$ [43]

It is noteworthy the fact that the daughter nuclei involved in a double beta process are stable against $\beta^{+}$transitions. Therefore $\chi_{1}$ is to be determined either using information about the half life of a $\beta^{+}$emitter lying close, in the nuclide chart, to the daughter nucleus under consideration or by fitting the data for a ( $\mathrm{p}, \mathrm{n}$ ) reaction having the daughter as a residual nucleus. As we already mentioned, throughout this work, the ratio $\chi / \chi_{1}$ is denoted, as usual, by $g_{p p}$.

The adopted procedure to fix the proton-neutron dipole interaction strengths is as follows. Whenever, in the intermediate odd-odd nucleus, the position of the GT resonance centroid is known, the $p h$ interaction strength is fixed so that the above mentioned data is reproduced. As shown in Table IV, for ${ }^{104} \mathrm{Ru}$ and ${ }^{110} \mathrm{Pd}$, the $\log f t$ values associated to the $\beta^{+} / E C$ and $\beta^{-}$transitions of the intermediate nuclei ${ }^{104} \mathrm{Rh}$ and ${ }^{110} \mathrm{Ag}$ respectively, are experimentally known. For these particular cases, $\chi$ and $g_{p p}$ are fixed by fitting the two mentioned experimental data. The $\log f t$ values were calculated by using
the following expression for $f t$ :

$$
\begin{equation*}
f t_{\mp}=\frac{6160}{\left[{ }_{l}\left\langle 1_{1}\right|\left|\beta^{ \pm}\right||0\rangle_{l} g_{A}\right]^{2}} . \tag{5.3}
\end{equation*}
$$

Here $\left|1_{1} M\right\rangle$ denotes the first dipole phonon state in the intermediate odd-odd nucleus while $|0\rangle$ is the pnQRPA ground state. The low index " $l$ " may take the value " $i$ " and " $f$ " depending whether the end state of the transition is characterizing the double beta mother or daughter nucleus. Therefore $l=f$ is associated to single $\beta^{-}$transition, while $l=i$ to the $\beta^{+} / E C$ process. We chose $g_{A}=1.0$ in order to take account of the effect of distant states responsible for the "missing strength" in the giant GT resonance [10].

For ${ }^{100} \mathrm{Tc},{ }^{116} \mathrm{In}$ and ${ }^{128} \mathrm{I}$ only the $\log f t$ values for their $\beta^{-}$decay are known. For these situations $\chi$ was fixed in order to get the right position for the GT resonance while $g_{p p}$ by fitting the $\log f t$ value characterizing the $\beta^{-}$decay of the corresponding intermediate nucleus. Regarding ${ }^{130} \mathrm{I}$ and ${ }^{134} \mathrm{Cs}$ we assumed the same $\log f t$ value as for ${ }^{128} \mathrm{I}$. For ${ }^{48} \mathrm{Ca}$, we considered $\chi$ and $g_{p p}$ as given by Eq.(5.2). To see the effect of $g_{p p}$ on $M_{G T}$ we repeated the calculations by keeping the same $\chi$ as before but taking $g_{p p}=0$. It seems that fixing $\chi$ as to reproduce the GT resonance centroid and taking $g_{p p}=0$ yields a better agreement with the experimental data. For ${ }^{96} \mathrm{Zr}, \chi$ was fixed by fitting the energy for the GT resonance centroid, while $g_{p p}$ was taken as required by Eq.5.2.

It is worth mentioning that both strength, $\chi$ and $\chi_{1}$, have been fixed by the calculations dealing with the ground to ground transition.

Concluding, the calculation of the transition rate $0^{+} \rightarrow 2^{+}$is free of any adjustable parameter
Having the RPA states defined, the GT amplitude has been calculated by means of Eq.(2.6), while the half life with Eq.(2.2). The Fermi integral for the transition $0^{+} \rightarrow 2^{+}$, denoted by $G_{02}$, was computed by using the analytical result given in Ref. [10].

### 5.1.5 Results for double beta decay

| Nucleus | $\begin{gathered} Q_{\beta \beta}^{2^{+}} \\ {\left[m_{e} c^{2}\right]} \end{gathered}$ | $\begin{gathered} \Delta E_{2} \\ {[\mathrm{MeV}]} \end{gathered}$ | $\begin{gathered} \left\|M_{G T}^{(0 \rightarrow 0)}\right\| \\ {\left[\mathrm{MeV}^{-1}\right]} \end{gathered}$ | $\begin{aligned} & \left\|M_{G T}^{(0 \rightarrow 2)}\right\| \\ & {\left[\mathrm{MeV}^{-3}\right]} \\ & \hline \end{aligned}$ | $T_{1 / 2}^{(0 \rightarrow 2)}[y r]$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  | present | Exp. | Ref. [26] |
| ${ }^{48} \mathrm{Ca}$ | 6.432 | 2.473 | 0.043 | $0.901 \cdot 10^{-3}$ | $1.72 \cdot 10^{24}$ |  |  |
| ${ }^{96} \mathrm{Zr}$ | 5.033 | 2.913 | 0.113 | $0.834 \cdot 10^{-3}$ | $2.27 \cdot 10^{25}$ | > 7.9.10 ${ }^{19}$ | 4.8.10 ${ }^{21}$ |
| ${ }^{100} \mathrm{Mo}$ | 4.874 | 1.756 | 0.305 | $0.136 \cdot 10^{-2}$ | $1.21 \cdot 10^{25}$ | $>1.6 \cdot 10^{21}$ | $3.9 \cdot 10^{24}$ |
|  |  |  |  |  |  |  | ${ }^{\text {a) }} 2.5 \cdot 10^{25}$ |
|  |  |  |  |  |  |  | ${ }^{\text {b) }} 1.2 \cdot 10^{26}$ |
| ${ }^{104} \mathrm{Ru}$ | 1.456 | 0.883 | 0.781 | 0.028 | $6.2 \cdot 10^{28}$ |  |  |
| ${ }^{110} \mathrm{Pd}$ | 2.646 | 1.182 | 0.263 | 0.050 | $1.48 \cdot 10^{25}$ |  |  |
| ${ }^{116} \mathrm{Cd}$ | 2.967 | 1.269 | 0.116 | 0.507•10 ${ }^{-2}$ | $3.4 \cdot 10^{26}$ | $>2.3 \cdot 10^{21}$ | $1.1 \cdot 10^{24}$ |
| ${ }^{128} \mathrm{Te}$ | 0.836 | 1.305 | 0.090 | $0.229 \cdot 10^{-2}$ | $4.7 \cdot 10^{33}$ | $>4.7 \cdot 10^{21}$ | $1.6 \cdot 10^{30}$ |
| ${ }^{130} \mathrm{Te}$ | 3.902 | 2.358 | 0.055 | $0.620 \cdot 10^{-3}$ | $6.94 \cdot 10^{26}$ | $>4.5 \cdot 10^{21}$ | $2.7 \cdot 10^{23}$ |
| ${ }^{134} \mathrm{Xe}$ | 0.460 | 0.806 | 0.039 | $0.621 \cdot 10^{-2}$ | $5.29 \cdot 10^{35}$ |  |  |
| ${ }^{136} \mathrm{Xe}$ | 3.251 | 1.518 | 0.039 | $0.249 \cdot 10^{-2}$ | $3.88 \cdot 10^{26}$ |  | $2.0 \cdot 10^{24}$ |

Table 5: The GT transition amplitudes and the half lives of the double beta decay $0^{+} \rightarrow 2^{+}$are given. Also, the $Q$ values are given in units of $m_{e} c^{2} . \Delta E_{2}$ is the energy shift defined in the text. For comparison, we give also the available experimental results as well as some theoretical predictions obtained with other formalisms: ${ }^{a), b}$ Ref.[29], ${ }^{c}$ ) Ref.[27], ${ }^{(), e)}[29]$ for different nuclear deformations, $\beta=0.28$ and $\beta=0.19$, respectively. The $M_{G T}$ values for the ground to ground transitions are also listed. For ${ }^{100}$ Mo we mention the result of Ref. ([29]) obtained with an $\mathrm{SU}(3)$ deformed single particle basis ${ }^{a)}$ and with a spherical basis ${ }^{b}$.

The final results are collected in Table V. Therein one may find also the available experimental data as well as some theoretical results obtained with other approaches. One notices that the half life is influenced by both the phase space integral (through the Q -value) and the single particle properties
which determine the transition amplitude. Indeed, for ${ }^{128} \mathrm{Te}$ and ${ }^{134} \mathrm{Xe}$ the small Q -value causes a very large half life, while in ${ }^{48} \mathrm{Ca}$ the opposite situation is met. By contrary, the Q value of ${ }^{110} \mathrm{Pd}$ is about the same as for ${ }^{76} \mathrm{Ge}$ but, due to the specific single particle and pairing properties of the orbits participating coherently to the process, the half life for the former case is more than three orders of magnitude less than in the later situation.

In order to isolate the deformation effect on the process half life we repeated the calculations in the spherical limit, i.e. $d \rightarrow 0$. In this limit the single particle energies coincide with the spherical shell model energies. The process of going to the spherical limit alters the pairing properties as well as the energies of the quadrupole collective states $2^{+}$, in mother and daughter nuclei. We modified the the strengths for pairing and $Q . Q$ interactions such that the pairing gaps and the $2^{+}$energies are the same as in the deformed picture. Also we preserved the dimension for single particle space for both protons and neutrons. For illustration we give here the result for the case of ${ }^{100} \mathrm{Mo}$ where the half life for the deformed situation, $1.21 \times 10^{25}$, becomes in the spherical limit equal to $0.46 \times 10^{24}$. Thus, one may conclude that the nuclear deformation enhances the decay half life. The same effect of deformation on the GT matrix elements was pointed out by Zamick and Auerbach in Ref.[47]. Indeed, they calculated the GT transition matrix elements for the neutrino capture $\nu_{\mu}+{ }^{12} C \rightarrow{ }^{12} N+\mu^{-}$using different structures for the ground states of ${ }^{12} C$ and ${ }^{12} N$ : a) spherical ground states; b) asymptotic limits of the wave functions and 3) deformed states with an intermediate deformation of $\delta=-0.3$. The results for the transition rate were $\frac{16}{3}, 0$ and 0.987 , respectively. Similar results are obtained also for the spin M1 transitions in ${ }^{12} C$. The ratio between the transition rates obtained with spherical and deformed basis explains the factor of 5 overestimate in the calculations of Ref.[48], where a spherical basis is used.

It is worth mentioning the good agreement between our prediction for ${ }^{100}$ Mo and that of Ref.[29] obtained with a deformed $\mathrm{SU}(3)$ single particle basis.

The transition matrix elements reported in Refs.[26, 27] are larger than those given here. The discrepancies are caused by the differences between the two approaches: a) In the quoted references one uses a spherical single particle basis, while here a deformed one is considered; b) The single particle energies used there are Woods-Saxon energies adjusted so that the quasiparticle spectrum in the odd-odd system be realistically described. We recall that the spherical limit of our model provides spherical shell model single particle energies. Also, the single particle spaces are different in the two formalisms; c) The higher RPA approach from Ref.[26] is the multiple commutator method (MCM) applied to the $p n Q R P A$ bosons or, alternatively [27], the renormalized pnQRPA bosons. A detailed comparison of the boson expansion formalism and MCM have been performed in Ref. [24]. It is a difficult task to make explicite the quantitative effect brought by the factors a), b), c) which, as a matter of fact, is beyond the scope of the present paper. However, concerning the sources a) and c) for the deviations one could draw some qualitative conclusions. Indeed, as we have already seen before, the nuclear deformation decreases the transition matrix element and consequently enhances the process half life. The MCM and boson expansion approaches provide different expressions for the terms which are cubic in bosons, involved in the transition operator. Indeed, the coefficients of these terms given by $M C M$ are cubic in the forward amplitudes $(X)$, while in the boson expansion formalism the expansion coefficients of the mentioned terms are at most quadratic in the amplitudes $X$. One expects, therefore, that $M C M$ provides larger matrix elements for these terms which results in having a shorter half life. Thus, the effects caused by the factors a) and c) are consistent with the sign of the discrepancies of results corresponding to the two approaches.

To have a reference value for the matrix elements associated to the transition $0^{+} \rightarrow 2^{+}$, in Table II are listed also the $M_{G T}$ values for the ground to ground transitions [20]. The ratio of the transition $0^{+} \rightarrow 0^{+}$and $0^{+} \rightarrow 2^{+}$matrix elements is quite large for ${ }^{76} \mathrm{Ge}(398),{ }^{100} \mathrm{Mo}(224)$ and ${ }^{96} \mathrm{Zr}$ (136) but small for ${ }^{110} \mathrm{Pd}(5.26)$ and ${ }^{134} \mathrm{Xe}$ (6.3). However, these ratios are not directly reflected in the half lives, since the phase space factors for the two transitions are very different from each other and, moreover, the differences depend on the atomic mass of the emitter.

The composing terms of the transition amplitude are suggestively represented in Fig.8. The term corresponding to Fig. 8 d) has a negligible contribution and, therefore, has been ignored. The terms
corresponding to the panels a), b) and c) of Fig. 8 are denoted by $M_{G T}^{(1)}, M_{G T}^{(2)}, M_{G T}^{(3)}$ respectively. In order to see the relative contribution of the three terms to the total amplitude, in Table VI we give, for illustration, the partial contribution for two nuclei. We see that for the cases listed in Table VI,

| Nucleus | $M_{G T}^{(0 \rightarrow 2)} \times 10^{6}$ <br> $\left[\mathrm{Mev}^{-3}\right]$ | $M_{G T}^{(1)} \times 10^{6}$ <br> $\left[\mathrm{Mev}^{-3}\right]$ | $M_{G T}^{(2)} \times 10^{6}$ <br> $\left[\mathrm{Mev}^{-3}\right]$ | $M_{G T}^{(3)} \times 10^{6}$ <br> $\left[\mathrm{Mev}^{-3}\right]$ |
| ---: | ---: | ---: | ---: | ---: |
| 48 <br> Ca | 901.100 | 900.000 | 0.07 | -0.004 |
| ${ }^{104} \mathrm{Ru}$ | 28281.530 | 26110.000 | 1583.810 | 587.72 |

Table 6: The values of the partial Gamow-Teller transition amplitudes $M_{G T}^{(1)}, M_{G T}^{(2)}, M_{G T}^{(3)}$, are given for some of the nuclei studied in the present paper. Their sum is denoted by $M_{G T}^{(0 \rightarrow 2)}$.
the term $M_{G T}^{(1)}$ prevails over the other ones. There are some cases not listed here where $M_{G T}^{2}$ is the dominant partial amplitude. None for the cases studied in our publications has $M_{G T}^{(3)}$ as a dominant term. The leading contributions coming from $M_{G T}^{(1)}$ and $M_{G T}^{(2)}$ have opposite sign. There are however two exceptions, the cases of ${ }^{104} \mathrm{Ru}$ and ${ }^{154} \mathrm{Sm}$, where the two contributions add coherently.

## 6 Conclusions

In the previous sections we presented the formalism as well as the numerical results for the two neutrino double beta decay to the collective excited state $2^{+}$. The Gamow-Teller transition rate has been calculated within a boson expansion formalism which is essentially a higher random phase approximation approach. The single particle basis is generated through an angular momentum projection procedure from a deformed set of states. The projected basis depends on a real parameter $d$ which simulates the nuclear deformation. In the limit of $d \rightarrow 0$ the spherical shell model basis is obtained while for $d$ different of zero, the single particle energies depend on the deformation parameter in a similar manner as the energies predicted by the Nilsson model. Due to these features the present formalism is able to describe in an unified fashion the spherical and deformed nuclei. In our previous publications we treated various situations when the mother and daughter had different deformations, in the context of the ground to ground double beta transition. We have seen that deformation causes a fragmentation of the single beta decays strength among the pnQRPA states. One expects that for the transition $0^{+} \rightarrow 2^{+}$the nuclear deformation is even more important. This can be understood even at the first glance since the larger the deformation of the daughter nucleus the lower the energy of the first $2^{+}$state. Consequently, the Q value is expected to be larger.

It is worth noticing that during the transition $0^{+} \rightarrow 2^{+}$several symmetries might be broken. Indeed, the second leg of the transition connects a magnetic state $1^{+}$from the intermediate odd-odd nucleus to an electric state $2^{+}$in the daughter nucleus. Among the nuclei considered in the present work there are situations when the mother nucleus is spherical while the daughter is a quadrupole deformed system. Moreover, in the case of ${ }^{160} \mathrm{Gd}$ decay there are suspicions that the mother has not a good space reflection symmetry [31] while the daughter satisfies this symmetry. Since the GT transition operator involves quadrupole phonon operators it may excite states whose isospin is different from that characterizing the mother ground state by $\Delta T=1,2$. The isospin mixing is also favored by the inclusion of the $p p$ interaction. On the other hand each symmetry breaking causes a new nuclear phase with specific properties. To our knowledge it is still an open question how these symmetry breaking are reflected in the decay rate. On this line, the results of the present work suggest to what direction the decay rate is modified by the nuclear deformation.

Concerning the quantitative description, the results presented in Table V reveal the following features. There are five nuclei whose half lives fall in the range accessible to experiment. These are: ${ }^{48} \mathrm{Ca},{ }^{96} \mathrm{Zr},{ }^{100} \mathrm{Mo},{ }^{110} \mathrm{Pd}$. Comparing with the results obtained by Toivanen and Suhonen $[26]$ or Civitarese and Suhonen [28], the half lives obtained in the present work are larger. The reason is that we use a deformed single particle basis while the quoted authors use a spherical one. The agreement
we obtain for ${ }^{100} \mathrm{Mo}$ with the calculations from Ref.[29], where a deformed $\mathrm{SU}(3)$ basis is used, support the above statement.

It is worth mentioning that the double beta transitions to excited states have been considered by several authors in the past, but the calculations emphasized the role of the transition operator and some specific selection rules. Many calculations regarded the neutrinoless process. Thus, in Ref.[49] it was shown that the neutrinoless transition to the excited $0^{+}$for medium heavy nuclei might be characterized by matrix elements which are larger than that of ground to ground transition and that happens since in the first transition, the change of the $K$ quantum number is less. In Ref.[50] it has been stated that the $0^{+} \rightarrow 2^{+}$matrix element depends on the left-right current coupling and not on the neutrino mass. This could provide a way of fixing the strength of the left-right coupling if the transition matrix element is experimentally known. However, according to the calculations of Haxton et al. [8], the matrix element is strongly suppressed and, therefore, the mentioned method of fixing the coupling parameter would not be reliable. Although the transition operator might have a complex structure, many calculations have been performed with the approximate interaction $[\sigma(1) \times \sigma(2)]^{\lambda=2} t_{+}(1) t_{+}(2)$ in order to test some selection rules. Thus, this interaction was used in Ref.[51] for the transition $0^{+} \rightarrow 2^{+}$of ${ }^{48} \mathrm{Ca}$, using a single $j$ calculation. It has been proved that the matrix element for this transition is suppressed due to the signature selection rules. Actually, this result confirms the feature of suppression for the $0^{+} \rightarrow 2^{+}$double beta transition matrix element pointed out by Vergados [52] and Haxton et al. [8].

The transition to $0_{1}^{+}$was examined for $A=76,82,100,136$ nuclei by assuming light and heavy Majorana neutrino exchange mechanism and trilinear R-parity violation. We recall that R parity is a discrete multiplicative symmetry defined as $R_{p}=(-1)^{3 B+L+2 S}$, where $\mathrm{S}, \mathrm{B}$ and L are the spin, the baryon and the lepton quantum number. Thus $R_{p}=+1$ for Standard Model particles and $R_{p}=-1$ for superpartners. Higher RPA as well as renormalization effects for the nuclear matrix elements were included [53].

Here we show that the transition $0^{+} \rightarrow 2^{+}$in a $2 \nu \beta \beta$ process is allowed by renormalizing the GT transition operator with some higher RPA corrections which results in making the matrix elements from Eq.(4.17) non-vanishing. Generally speaking, transitions to the excited states are suppressed due to the reduced $Q_{\beta \beta}$ value. However, this restriction could, in some cases, be compensated by a possible lower background due to the coincindence of the $\beta$ particles with the $\gamma$ from the excited final state. Indeed, our calculations pointed out that, for some nuclei, the nuclear matrix elements for the transitions to the state $2^{+}$are comparable to those charcaterizing the ground to ground transition.

The calculated $M_{G T}$ values of the present work are smaller than those from Ref.[26] obtained with a spherical single particle basis, which agrees with the earlier calculations of Zamick and Auerbach for ${ }^{12} C$, showing that the nuclear deformation suppresses the GT matrix elements.

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