# No cross-couplings among different spin-two fields intermediated by a massless $p$-form 

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#### Abstract

Under the general hypotheses of locality, smoothness of interactions in the coupling constant, Poincaré invariance, Lorentz covariance, and preservation of the number of derivatives on each field, we review the problem of the construction of cross-couplings of one or several spin-two fields to a massless $p$-form.


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## 1 Chronology

During the last 10 years the problem of constructing consistent interactions among different spin-two fields has been approached from a BRST cohomological perspective in several papers:

- N. Boulanger, T. Damour, L. Gualtieri and M. Henneaux, Inconsistency of interacting, multigraviton theories, Nucl. Phys. B597 (2001) 127-171 [arXiv:hep-th/0007220]
- C. Bizdadea, E. M. Cioroianu, A. C. Lungu and S. O. Saliu, No multi-graviton theories in the presence of a Dirac field, J. High Energy Phys. JHEP 0502 (2005) 016 [arXiv:0704.2321v1[hepth]]
- C. Bizdadea, E. M. Cioroianu, D. Cornea, S. O. Saliu and S. C. Sararu, No interactions for a collection of spin-two fields intermediated by a massive Rarita-Schwinger field, Eur. Phys. J. C48 (2006) 265-289 [arXiv:0704.2334v1[hep-th]].

All these no-go results have been deduced under the general hypotheses: space-time locality, smoothness in the coupling constant, (background) Lorentz covariance, Poincaré invariance (i.e. we do not allow explicit dependence on the spacetime coordinates) and preservation of the number of derivatives on each field (such that the differential order of the deformed field equations is preserved with respect to the free model)-derivative order assumption.

It is nevertheless known that the relaxation of the derivative order condition may lead to exotic couplings for one or a collection of spin-two fields, which are no longer mastered by General Relativity.

- N. Boulanger and L. Gualtieri, An exotic theory of massless spin-two fields in three dimensions, Class. Quant. Grav. 18 (2001) 1485-1502 [arXiv:hep-th/0012003]

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## 2 Overview

In this talk we approach the folowing problems:
i) the construction of the couplings among a massless vector field and one spin-two field;
ii) the derivation of the interactions among a massless vector field and several spin-two fields;
iii) the generalization of the previous results to the case of couplings between one or several gravitons and an arbitrary massless $p$-form gauge field.

We use the method of consistent deformations of the generator of the Lagrangian BRST symmetry (known as the solution of the master equation) by means of specific cohomological techniques, relying on local BRST cohomology.

The general hypotheses are the same like in the above.
The talk is based on the paper

- C. Bizdadea, E. M. Cioroianu, D. Cornea, E. Diaconu, S. O. Saliu and S. C. Sararu, Interactions for a collection of spin-two fields intermediated by a massless p-form, Nucl. Phys. B794 (2008) 442-494 [arXiv:0705.3210v2[hep-th]].


## 3 Consistent interactions between the spin-two field and a massless vector field

### 3.1 BRST symmetry of the free model

Our starting point is represented by a free Lagrangian action, written as the sum between the linearized Hilbert-Einstein action (also known as the Pauli-Fierz action) and Maxwell's action in $D>2$ spacetime dimensions

$$
\begin{align*}
S_{0}^{\mathrm{L}}\left[h_{\mu \nu}, V_{\mu}\right]= & \int d^{D} x\left[-\frac{1}{2}\left(\partial_{\mu} h_{\nu \rho}\right) \partial^{\mu} h^{\nu \rho}+\left(\partial_{\mu} h^{\mu \rho}\right) \partial^{\nu} h_{\nu \rho}\right. \\
& \left.-\left(\partial_{\mu} h\right) \partial_{\nu} h^{\nu \mu}+\frac{1}{2}\left(\partial_{\mu} h\right) \partial^{\mu} h-\frac{1}{4} F_{\mu \nu} F^{\mu \nu}\right]  \tag{1}\\
\equiv & \int d^{D} x\left(\mathcal{L}_{0}^{(\mathrm{PF})}+\mathcal{L}_{0}^{(\text {vect })}\right)
\end{align*}
$$

The restriction $D>2$ is required by the spin-two field action, which is known to reduce to a total derivative in $D=2$. Throughout the paper we work with the flat metric of 'mostly plus' signature, $\sigma_{\mu \nu}=(-+\ldots+)$. In the above $h$ denotes the trace of the Pauli-Fierz field, $h=\sigma_{\mu \nu} h^{\mu \nu}$, and $F_{\mu \nu}$ represents the Abelian field-strength of the massless vector field $\left(F_{\mu \nu} \equiv \partial_{[\mu} V_{\nu]}\right)$. The theory described by action (1) possesses an Abelian and irreducible generating set of gauge transformations

$$
\begin{equation*}
\delta_{\epsilon} h_{\mu \nu}=\partial_{\mu} \epsilon_{\nu}+\partial_{\nu} \epsilon_{\mu} \equiv \partial_{(\mu} \epsilon_{\nu)}, \quad \delta_{\epsilon} V_{\mu}=\partial_{\mu} \epsilon \tag{2}
\end{equation*}
$$

with $\epsilon_{\mu}$ and $\epsilon$ bosonic gauge parameters. The notation $[\mu \ldots \nu]$ (or $(\mu \ldots \nu)$ ) signifies antisymmetry (or symmetry) with respect to all indices between brackets without normalization factors (i.e., the independent terms appear only once and are not multiplied by overall numerical factors).

In order to construct the BRST symmetry for action (1), it is necessary to introduce the field/ghost and antifield spectra

$$
\begin{align*}
\Phi^{\alpha_{0}} & =\left(h_{\mu \nu}, V_{\mu}\right), \quad \Phi_{\alpha_{0}}^{*}=\left(h^{* \mu \nu}, V^{* \mu}\right),  \tag{3}\\
\eta_{\alpha_{1}} & =\left(\eta_{\mu}, \eta\right), \quad \eta^{* \alpha_{1}}=\left(\eta^{* \mu}, \eta^{*}\right) . \tag{4}
\end{align*}
$$

The fermionic ghosts $\eta_{\alpha_{1}}$ are associated with the gauge parameters $\epsilon_{\alpha_{1}}=\left\{\epsilon_{\mu}, \epsilon\right\}$ respectively and the star variables represent the antifields of the corresponding fields/ghosts. (According to the standard
rule of the BRST method, the Grassmann parity of a given antifield is opposite to that of the corresponding field/ghost.) Since the gauge generators are field-independent and irreducible, it follows that the BRST differential decomposes into

$$
\begin{equation*}
s=\delta+\gamma \tag{5}
\end{equation*}
$$

where $\delta$ is the Koszul-Tate differential and $\gamma$ denotes the exterior longitudinal derivative. The KoszulTate differential is graded in terms of the antighost number (agh, agh $(\delta)=-1$, agh $(\gamma)=0)$ and enforces a resolution of the algebra of smooth functions defined on the stationary surface of field equations for action $(1), C^{\infty}(\Sigma), \Sigma: \delta S_{0}^{\mathrm{L}} / \delta \Phi^{\alpha_{0}}=0$. The exterior longitudinal derivative is graded in terms of the pure ghost number ( $\operatorname{pgh}, \operatorname{pgh}(\gamma)=1, \operatorname{pgh}(\delta)=0)$ and is correlated with the original gauge symmetry via its cohomology in pure ghost number zero computed in $C^{\infty}(\Sigma)$, which is isomorphic to the algebra of physical observables for this free theory. These two degrees of the BRST generators are valued as

$$
\begin{align*}
& \operatorname{agh}\left(\Phi^{\alpha_{0}}\right)=\operatorname{agh}\left(\eta_{\alpha_{1}}\right)=0, \quad \operatorname{agh}\left(\Phi_{\alpha_{0}}^{*}\right)=1, \quad \operatorname{agh}\left(\eta^{* \alpha_{1}}\right)=2  \tag{6}\\
& \operatorname{pgh}\left(\Phi^{\alpha_{0}}\right)=0, \quad \operatorname{pgh}\left(\eta_{\alpha_{1}}\right)=1, \quad \operatorname{pgh}\left(\Phi_{\alpha_{0}}^{*}\right)=\operatorname{pgh}\left(\eta^{* \alpha_{1}}\right)=0 \tag{7}
\end{align*}
$$

The overall degree that grades the BRST complex is named ghost number (gh) and is defined like the difference between the pure ghost number and the antighost number, such that gh $(s)=\operatorname{gh}(\delta)=$ $\operatorname{gh}(\gamma)=1$. The actions of the operators $\delta$ and $\gamma$ (taken to act as right differentials) on the BRST generators read as

$$
\begin{align*}
\delta h^{* \mu \nu} & =2 H^{\mu \nu}, \quad \delta V^{* \mu}=-\partial_{\nu} F^{\nu \mu}  \tag{8}\\
\delta \eta^{* \mu} & =-2 \partial_{\nu} h^{* \nu \mu}, \quad \delta \eta^{*}=-\partial_{\mu} V^{* \mu}  \tag{9}\\
\delta \Phi^{\alpha_{0}} & =0, \quad \delta \eta_{\alpha_{1}}=0  \tag{10}\\
\gamma \Phi_{\alpha_{0}}^{*} & =0, \quad \quad \gamma \eta^{* \alpha_{1}}=0  \tag{11}\\
\gamma h_{\mu \nu} & =\partial_{(\mu} \eta_{\nu)}, \quad \gamma V_{\mu}=\partial_{\mu} \eta  \tag{12}\\
\gamma \eta_{\mu} & =0, \quad \gamma \eta=0 \tag{13}
\end{align*}
$$

In the above $H^{\mu \nu}$ is the linearized Einstein tensor

$$
\begin{equation*}
H^{\mu \nu}=K^{\mu \nu}-\frac{1}{2} \sigma^{\mu \nu} K \tag{14}
\end{equation*}
$$

with $K^{\mu \nu}$ and $K$ the linearized Ricci tensor and the linearized scalar curvature respectively, both obtained from the linearized Riemann tensor

$$
\begin{equation*}
K_{\mu \nu \mid \alpha \beta}=-\frac{1}{2}\left(\partial_{\mu} \partial_{\alpha} h_{\nu \beta}+\partial_{\nu} \partial_{\beta} h_{\mu \alpha}-\partial_{\nu} \partial_{\alpha} h_{\mu \beta}-\partial_{\mu} \partial_{\beta} h_{\nu \alpha}\right) \tag{15}
\end{equation*}
$$

from its trace and double trace respectively

$$
\begin{equation*}
K_{\mu \alpha}=\sigma^{\nu \beta} K_{\mu \nu \mid \alpha \beta}, \quad K=\sigma^{\mu \alpha} \sigma^{\nu \beta} K_{\mu \nu \mid \alpha \beta} \tag{16}
\end{equation*}
$$

The BRST differential is known to have a canonical action in a structure named antibracket and denoted by the symbol $(),(s \cdot=(\cdot, \bar{S}))$, which is obtained by considering the fields/ghosts conjugated respectively to the corresponding antifields. The generator of the BRST symmetry is a bosonic functional of ghost number zero, which is solution to the classical master equation $(\bar{S}, \bar{S})=0$. The full solution to the master equation for the free model under study reads as

$$
\begin{equation*}
\bar{S}=S_{0}^{\mathrm{L}}\left[h_{\mu \nu}, V_{\mu}\right]+\int d^{D} x\left(h^{* \mu \nu} \partial_{(\mu} \eta_{\nu)}+V^{* \mu} \partial_{\mu} \eta\right) \tag{17}
\end{equation*}
$$

and encodes all the information on the gauge structure of the theory (1)-(2).

### 3.2 Brief review of the deformation procedure

We begin with a "free" gauge theory, described by a Lagrangian action $S_{0}^{\mathrm{L}}\left[\Phi^{\alpha_{0}}\right]$, invariant under some gauge transformations $\delta_{\epsilon} \Phi^{\alpha_{0}}=\bar{Z}_{\alpha_{1}}^{\alpha_{1}}{ }^{\alpha_{1}}$, i.e. $\frac{\delta S^{L}}{\delta \Phi^{\alpha}} \bar{Z}_{\alpha_{1}}^{\alpha_{0}}=0$, and consider the problem of constructing consistent interactions among the fields $\Phi^{\alpha_{0}}$ such that the couplings preserve the field spectrum and the original number of gauge symmetries. This matter is addressed by means of reformulating the problem of constructing consistent interactions as a deformation problem of the solution to the master equation corresponding to the "free" theory. Such a reformulation is possible due to the fact that the solution to the master equation contains all the information on the gauge structure of the theory. If an interacting gauge theory can be consistently constructed, then the solution $\bar{S}$ to the master equation associated with the "free" theory, $(\bar{S}, \bar{S})=0$, can be deformed into a solution $S$

$$
\begin{equation*}
\bar{S} \rightarrow S=\bar{S}+k S_{1}+k^{2} S_{2}+\cdots=\bar{S}+k \int d^{D} x a+k^{2} \int d^{D} x b+\cdots \tag{18}
\end{equation*}
$$

of the master equation for the deformed theory

$$
\begin{equation*}
(S, S)=0 \tag{19}
\end{equation*}
$$

such that both the ghost and antifield spectra of the initial theory are preserved. The projection of equation (19) on the various orders in the coupling constant $k$ leads to the equivalent tower of equations

$$
\begin{align*}
(\bar{S}, \bar{S}) & =0  \tag{20}\\
2\left(S_{1}, \bar{S}\right) & =0  \tag{21}\\
2\left(S_{2}, \bar{S}\right)+\left(S_{1}, S_{1}\right) & =0 \tag{22}
\end{align*}
$$

Equation (20) is fulfilled by hypothesis. The next equation requires that the first-order deformation of the solution to the master equation, $S_{1}$, is a co-cycle of the "free" BRST differential $s, s S_{1}=0$. However, only cohomologically nontrivial solutions to (21) should be taken into account, since the BRST-exact ones can be eliminated by some (in general nonlinear) field redefinitions. This means that $S_{1}$ pertains to the ghost number zero cohomological space of $s, H^{0}(s)$, which is nonempty because it is isomorphic to the space of physical observables of the "free" theory. It has been shown (by of the triviality of the antibracket map in the cohomology of the BRST differential) that there are no obstructions in finding solutions to the remaining equations, namely (22), etc. However, the resulting interactions may be nonlocal and there might even appear obstructions if one insists on their locality. The analysis of these obstructions can be done with the help of cohomological techniques.

### 3.3 Standard material: basic cohomologies

If we make the notation $S_{1}=\int d^{D} x a$, then equation (21), which controls the first-order deformation, takes the local form

$$
\begin{equation*}
s a=\partial_{\mu} m^{\mu}, \quad \operatorname{gh}(a)=0, \quad \varepsilon(a)=0, \tag{23}
\end{equation*}
$$

for some local current $m^{\mu}$.
In order to analyze equation (23) we develop $a$ according to the antighost number

$$
\begin{equation*}
a=\sum_{i=0}^{I} a_{i}, \quad \operatorname{agh}\left(a_{i}\right)=i, \quad \operatorname{gh}\left(a_{i}\right)=0, \quad \varepsilon\left(a_{i}\right)=0, \tag{24}
\end{equation*}
$$

and assume, without loss of generality, that decomposition (24) stops at some finite value of $I$. Replacing decomposition (24) into (23) and projecting it on the various values of the antighost number by means of (5), we obtain that (23) is equivalent with the tower of equations

$$
\begin{equation*}
\gamma a_{I}=0, \tag{25}
\end{equation*}
$$

$$
\begin{align*}
\delta a_{I}+\gamma a_{I-1} & =\partial_{\mu} m_{I-1}^{\mu}  \tag{26}\\
\delta a_{i}+\gamma a_{i-1} & =\partial_{\mu} m_{i-1}^{\mu}, \quad 1 \leq i \leq I-1 \tag{27}
\end{align*}
$$

where $\left(m_{i}^{\mu}\right)_{i=\overline{0, I-1}}$ are some local currents, with $\operatorname{agh}\left(m_{i}^{\mu}\right)=i$. In other words, the nontriviality of the first-order deformation $a$ is translated at its highest antighost number component into the requirement that $a_{I} \in H^{I}(\gamma)$, where $H^{I}(\gamma)$ denotes the cohomology of the exterior longitudinal derivative $\gamma$ in pure ghost number equal to $I$. So, in order to solve equation (25), we need to compute the cohomology of $\gamma, H(\gamma)$.

Using the results on the cohomology of $\gamma$ in the Pauli-Fierz sector as well as definitions (11)-(13), we can state that $H(\gamma)$ is generated on the one hand by $\Phi_{\alpha_{0}}^{*}, \eta^{* \alpha_{1}}, F_{\mu \nu}$, and $K_{\mu \nu \alpha \beta}$, together with their spacetime derivatives and, on the other hand, by the undifferentiated ghosts $\eta$ and $\eta_{\mu}$ as well as by their antisymmetric first-order derivatives $\partial_{[\mu} \eta_{\nu]}$. (The spacetime derivatives of $\eta$ are $\gamma$-exact, in agreement with the latter definition from (12), and the same is valid for the derivatives of $\eta_{\mu}$ of order two and higher.) So, the most general (and nontrivial) solution to (25) can be written, up to $\gamma$-exact contributions, as

$$
\begin{equation*}
a_{I}=\alpha_{I}\left(\left[F_{\mu \nu}\right],\left[K_{\mu \nu \rho \lambda}\right],\left[\Phi_{\alpha_{0}}^{*}\right],\left[\eta^{* \alpha_{1}}\right]\right) e^{I}\left(\eta, \eta_{\mu}, \partial_{[\mu} \eta_{\nu]}\right) \tag{28}
\end{equation*}
$$

where the notation $f([q])$ means that $f$ depends on $q$ and its derivatives up to a finite order, while $e^{I}$ denotes the elements of a basis in the space of polynomials with pure ghost number $I$ in $\eta, \eta_{\mu}$, and $\partial_{[\mu} \eta_{\nu]}$. The objects $\alpha_{I}$ (obviously nontrivial in $H^{0}(\gamma)$ ) were taken to have a finite antighost number and a bounded number of derivatives, and therefore they are polynomials in the antifields, in the linearized Riemann tensor $K_{\mu \nu \alpha \beta}$, and in the field-strength $F_{\mu \nu}$ as well as in their subsequent derivatives. They are required to fulfill the property agh $\left(\alpha_{I}\right)=I$ in order to ensure that the ghost number of $a_{I}$ is equal to zero. Due to their $\gamma$-closeness, $\gamma \alpha_{I}=0$, and to their polynomial character, $\alpha_{I}$ will be called invariant polynomials. In antighost number zero the invariant polynomials are polynomials in the linearized Riemann tensor, in the field-strength of the Abelian field, and in their derivatives.

Inserting (28) in (26), we obtain that a necessary (but not sufficient) condition for the existence of (nontrivial) solutions $a_{I-1}$ is that the invariant polynomials $\alpha_{I}$ are (nontrivial) objects from the local cohomology of the Koszul-Tate differential $H(\delta \mid d)$ in antighost number $I>0$ and in pure ghost number zero

$$
\begin{equation*}
\delta \alpha_{I}=\partial_{\mu} j_{I-1}^{\mu}, \quad \operatorname{agh}\left(j_{I-1}^{\mu}\right)=I-1, \quad \operatorname{pgh}\left(j_{I-1}^{\mu}\right)=0 \tag{29}
\end{equation*}
$$

Using the fact that the Cauchy order of the free theory under study is equal to two, the general results from literature, according to which the local cohomology of the Koszul-Tate differential in pure ghost number zero is trivial in antighost numbers strictly greater than its Cauchy order, ensure that

$$
\begin{equation*}
H_{J}(\delta \mid d)=0, \quad J>2 \tag{30}
\end{equation*}
$$

where $H_{J}(\delta \mid d)$ denotes the local cohomology of the Koszul-Tate differential in antighost number $J$ and in pure ghost number zero. It can be shown that any invariant polynomial that is trivial in $H_{J}(\delta \mid d)$ with $J \geq 2$ can be taken to be trivial also in $H_{J}^{\text {inv }}(\delta \mid d)$. $\left(H_{J}^{\text {inv }}(\delta \mid d)\right.$ denotes the invariant characteristic cohomology in antighost number $J$ - the local cohomology of the Koszul-Tate differential in the space of invariant polynomials.) Thus:

$$
\begin{equation*}
\left(\alpha_{J}=\delta b_{J+1}+\partial_{\mu} c_{J}^{\mu}, \operatorname{agh}\left(\alpha_{J}\right)=J \geq 2\right) \Rightarrow \alpha_{J}=\delta \beta_{J+1}+\partial_{\mu} \gamma_{J}^{\mu} \tag{31}
\end{equation*}
$$

with both $\beta_{J+1}$ and $\gamma_{J}^{\mu}$ invariant polynomials. Results (31) and (30) yield the conclusion that the invariant characteristic cohomology is trivial in antighost numbers strictly greater than two

$$
\begin{equation*}
H_{J}^{\mathrm{inv}}(\delta \mid d)=0, \quad J>2 \tag{32}
\end{equation*}
$$

Moreover, it can be proved that the spaces $H_{2}(\delta \mid d)$ and $H_{2}^{\mathrm{inv}}(\delta \mid d)$ are spanned by

$$
\begin{equation*}
H_{2}(\delta \mid d), H_{2}^{\mathrm{inv}}(\delta \mid d):\left(\eta^{*}, \eta^{* \mu}\right) \tag{33}
\end{equation*}
$$

In contrast to the groups $\left(H_{J}(\delta \mid d)\right)_{J \geq 2}$ and $\left(H_{J}^{\text {inv }}(\delta \mid d)\right)_{J \geq 2}$, which are finite-dimensional, the cohomology $H_{1}(\delta \mid d)$ in pure ghost number zero, known to be related to global symmetries and ordinary
conservation laws, is infinite-dimensional since the theory is free. Fortunately, it will not be needed in the sequel.

The previous results on $H(\delta \mid d)$ and $H^{\text {inv }}(\delta \mid d)$ in strictly positive antighost numbers are important because they control the obstructions of removing the antifields from the first-order deformation. Based on formulas (30)-(32), one can eliminate all the pieces of antighost number strictly greater than two from the nonintegrated density of the first-order deformation by adding only trivial terms. Consequently, one can take (without loss of nontrivial objects) $I \leq 2$ into the decomposition (24). In addition, the last representative reads as in (28), where the invariant polynomial is necessarily a nontrivial object from $H_{2}^{\text {inv }}(\delta \mid d)$ if $I=2$ and from $H_{1}(\delta \mid d)$ if $I=1$ respectively.

### 3.4 First-order deformation

With these ingredients at hand we find that the most general, nontrivial first-order deformation of the solution to the master equation corresponding to action (1) and to its gauge transformations (2), which complies with all the working hypotheses, is expressed by

$$
\begin{equation*}
S_{1}=S_{1}^{(\mathrm{PF})}+S_{1}^{(\mathrm{int})} \tag{34}
\end{equation*}
$$

where

$$
\begin{align*}
S_{1}^{(\mathrm{PF})}= & \int d^{D} x\left[\frac{1}{2} f \eta^{* \mu} \eta^{\nu} \partial_{[\mu} \eta_{\nu]}+f h^{* \mu \rho}\left(\left(\partial_{\rho} \eta^{\nu}\right) h_{\mu \nu}-\eta^{\nu} \partial_{[\mu} h_{\nu] \rho}\right)\right. \\
& \left.+f a_{0}^{(\mathrm{EH}-\mathrm{cubic})}-2 \Lambda h\right] \tag{35}
\end{align*}
$$

and

$$
\begin{align*}
S_{1}^{(\text {int })}= & \int d^{D} x\left\{y_{2}\left[h^{*} \eta+(D-2)\left(-V^{* \lambda} \eta_{\lambda}+V^{\lambda} \partial_{[\mu} h_{\lambda]}{ }^{\mu}\right)\right]\right. \\
& +y_{3} \delta_{3}^{D} \varepsilon_{\mu \nu \rho}\left(V^{* \mu} \partial^{[\nu} \eta^{\rho]}+F^{\lambda \mu} \partial^{[\nu} h_{\lambda}^{\rho]}\right)+p\left[\eta^{*} \eta_{\mu} \partial^{\mu} \eta\right. \\
& -\frac{1}{2} V^{* \mu}\left(V^{\nu} \partial_{[\mu} \eta_{\nu]}+2\left(\partial_{\nu} V_{\mu}\right) \eta^{\nu}-h_{\mu \nu} \partial^{\nu} \eta\right) \\
& \left.+\frac{1}{8} F^{\mu \nu}\left(2 \partial_{[\mu}\left(h_{\nu] \rho} V^{\rho}\right)+F_{\mu \nu} h-4 F_{\mu \rho} h^{\rho}{ }_{\nu}\right)\right] \\
& \left.+q_{1} \delta_{3}^{D} \varepsilon^{\mu \nu \lambda} V_{\mu} F_{\nu \lambda}+q_{2} \delta_{5}^{D} \varepsilon^{\mu \nu \lambda \alpha \beta} V_{\mu} F_{\nu \lambda} F_{\alpha \beta}\right\}, \tag{36}
\end{align*}
$$

with $\delta_{m}^{D}$ the Kronecker symbol. Thus, the first-order deformation of the solution to the master equation for the model under study is parameterized by seven independent, real constants, namely $f$ and $\Lambda$ corresponding to $S_{1}^{(\mathrm{PF})}$ together with $p, y_{2}, y_{3} \delta_{3}^{D}, q_{1} \delta_{3}^{D}$, and $q_{2} \delta_{5}^{D}$ associated with $S_{1}^{(\text {int })}$.

### 3.5 Higher-order deformations

The construction of the second-order deformation of the solution to the master equation is governed by equation (22). Replacing (34) into (22) we find that it leads to the equations

$$
\begin{align*}
p(p+f) & =0  \tag{37}\\
(2 p+f) y_{3} \delta_{3}^{D} & =0  \tag{38}\\
(2 p+f) y_{2} & =0 \tag{39}
\end{align*}
$$

for the constants that parameterizes the first order deformation. There are three relevant solutions to the above equations

$$
\begin{align*}
\text { Case I }: & p=-f \neq 0, \quad y_{2}=0=y_{3} \delta_{3}^{D}, \quad D>2,  \tag{40}\\
\text { Case II }: & p=f=0, \quad D=3, \tag{41}
\end{align*}
$$

$$
\begin{equation*}
\text { Case III }: p=f=0, \quad D>3, \tag{42}
\end{equation*}
$$

which require an individual treatment. In general, the consistency of the deformations of orders three, four, and higher may impose new restrictions upon the constants that parameterize the first-order deformation.

### 3.5.1 Case I - General Relativity

For the sake of simplicity we set $f=1$, so $p=-1$. Then, after long computations we find that in the first case the consistent interactions between a graviton and a vector field are described in all $D>2$ dimensions by the Lagrangian action

$$
\begin{align*}
& S^{\mathrm{L}(\mathrm{I})}\left[g_{\mu \nu}, \bar{V}_{\mu}\right]=\int d^{D} x\left[\frac{2}{k^{2}} \sqrt{-g}\left(R-2 k^{2} \Lambda\right)-\frac{1}{4} \sqrt{-g} g^{\mu \nu} g^{\rho \lambda} \bar{F}_{\mu \rho} \bar{F}_{\nu \lambda}\right. \\
& \left.+k\left(q_{1} \delta_{3}^{D} \varepsilon^{\mu_{1} \mu_{2} \mu_{3}} \bar{V}_{\mu_{1}} \bar{F}_{\mu_{2} \mu_{3}}+q_{2} \delta_{5}^{D} \varepsilon^{\mu_{1} \mu_{2} \mu_{3} \mu_{4} \mu_{5}} \bar{V}_{\mu_{1}} \bar{F}_{\mu_{2} \mu_{3}} \bar{F}_{\mu_{4} \mu_{5}}\right)\right] \tag{43}
\end{align*}
$$

that is invariant under the deformed gauge transformations

$$
\begin{equation*}
\delta_{\epsilon}^{(\mathrm{I})} g_{\mu \nu}=k \epsilon_{(\mu ; \nu)}, \quad \delta_{\epsilon}^{(\mathrm{I})} \bar{V}_{\mu}=\partial_{\mu} \epsilon+k\left(\partial_{\mu} \bar{\epsilon}^{\nu}\right) \bar{V}_{\nu}+k\left(\partial_{\nu} \bar{V}_{\mu}\right) \bar{\epsilon}^{\nu} \tag{44}
\end{equation*}
$$

where

$$
\begin{equation*}
g_{\mu \nu}=\sigma_{\mu \nu}+k h_{\mu \nu}, \quad \bar{V}_{\mu}=e_{\mu}^{a} V_{a} \tag{45}
\end{equation*}
$$

while $\epsilon_{\mu ; \nu}$ is the (full) covariant derivative of $\epsilon_{\mu}$ and $e_{\mu}^{a}$ is the inverse of vielbein $e_{a}^{\mu}$.
Our result follows as a consequence of applying a cohomological procedure based on the "free" BRST symmetry in the presence of a few natural assumptions. General covariance was not imposed a priori, but was gained in a natural way from the cohomological setting developed here under the previously mentioned hypotheses.

In conclusion, the first case outputs the formulas (43)-(44) which turn out to describe nothing but the standard graviton-vector interactions from General Relativity.

### 3.5.2 Case II - new solutions in $D=3$

In this situation we obtain two subcases, but only one is relevant. In the relevant subcase, the consistency of the deformation procedure imposes that

$$
\begin{equation*}
q_{2} \delta_{5}^{D}=y_{2}=0 \tag{46}
\end{equation*}
$$

such the Lagrangian action of the coupled model is given by (for $y_{3}=1$ )

$$
\begin{align*}
S^{\mathrm{L}(\mathrm{II})}\left[h_{\mu \nu}, V_{\mu}\right]= & \int d^{3} x\left[\mathcal{L}_{0}^{(\mathrm{PF})}-\frac{1}{4} F_{\mu \nu} F^{\mu \nu}-2 k \Lambda h\right. \\
& \left.-k F^{\mu \nu} \varepsilon_{\mu \nu \rho} \partial^{[\theta} h_{\theta}^{\rho]}+2 k^{2}\left(\partial^{[\mu} h_{\mu}^{\rho]}\right) \partial_{[\nu} h_{\rho]}{ }^{\nu}\right] \tag{47}
\end{align*}
$$

where $\mathcal{L}_{0}^{(\mathrm{PF})}$ is the Pauli-Fierz Lagrangian and $\Lambda$ is the cosmological constant, while the gauge symmetries of (47) read as

$$
\begin{equation*}
\delta_{\epsilon}^{(\mathrm{II})} h_{\mu \nu}=\partial_{(\mu} \epsilon_{\nu)}, \quad \delta_{\epsilon}^{(\mathrm{II})} V_{\mu}=\partial_{\mu} \epsilon+k \varepsilon_{\mu \nu \rho} \partial^{[\nu} \epsilon^{\rho]} \tag{48}
\end{equation*}
$$

In conclusion, this subcase yields another possibility to establish nontrivial couplings between the Pauli-Fierz field and a vector field. It is complementary to case I (General Relativity) due to the consistency conditions (37)-(39) and is valid only in $D=3$.

### 3.5.3 Case III - nothing new

The case III brings no new information on the possible couplings between a spin-two field and a massless one-form.

## 4 No cross-couplings in multi-graviton theories intermediated by a vector field

### 4.1 First-order deformation. Consistency conditions

### 4.1.1 Generalities

We start now from a finite sum of Pauli-Fierz actions and a single Maxwell action in $D>2$

$$
\begin{align*}
S_{0}^{\mathrm{L}}\left[h_{\mu \nu}^{A}, V_{\mu}\right]= & \int d^{D} x\left[-\frac{1}{2}\left(\partial_{\mu} h_{\nu \rho}^{A}\right) \partial^{\mu} h_{A}^{\nu \rho}+\left(\partial_{\mu} h_{A}^{\mu \rho}\right) \partial^{\nu} h_{\nu \rho}^{A}\right. \\
& \left.-\left(\partial_{\mu} h^{A}\right) \partial_{\nu} h_{A}^{\nu \mu}+\frac{1}{2}\left(\partial_{\mu} h^{A}\right) \partial^{\mu} h_{A}-\frac{1}{4} F_{\mu \nu} F^{\mu \nu}\right], \tag{49}
\end{align*}
$$

where $h_{A}$ is the trace of the Pauli-Fierz field $h_{A}^{\mu \nu}\left(h_{A}=\sigma_{\mu \nu} h_{A}^{\mu \nu}\right)$, with $A=\overline{1, n}$ and $n>1$. The collection indices $A, B$, etc., are raised and lowered with a quadratic form $k_{A B}$ that determines a positively-defined metric in the internal space. It can always be normalized to $\delta_{A B}$ by a simple linear field redefinition, so from now on we take $k_{A B}=\delta_{A B}$ and re-write (49) as

$$
\begin{equation*}
S_{0}^{\mathrm{L}}\left[h_{\mu \nu}^{A}, V_{\mu}\right]=\int d^{D} x\left[\sum_{A=1}^{n} \mathcal{L}_{0}^{(\mathrm{PF})}\left(h_{\mu \nu}^{A}, \partial_{\lambda} h_{\mu \nu}^{A}\right)+\mathcal{L}_{0}^{(\text {vect })}\right] \tag{50}
\end{equation*}
$$

where $\mathcal{L}_{0}^{(\mathrm{PF})}\left(h_{\mu \nu}^{A}, \partial_{\lambda} h_{\mu \nu}^{A}\right)$ is the Pauli-Fierz Lagrangian for the graviton $A$. Action (49) is invariant under the gauge transformations

$$
\begin{equation*}
\delta_{\epsilon} h_{\mu \nu}^{A}=\partial_{(\mu} \epsilon_{\nu)}^{A}, \quad \delta_{\epsilon} V_{\mu}=\partial_{\mu} \epsilon \tag{51}
\end{equation*}
$$

The BRST complex comprises the fields, ghosts, and antifields

$$
\begin{align*}
& \hat{\Phi}^{\alpha_{0}}=\left(h_{\mu \nu}^{A}, V_{\mu}\right), \quad \hat{\eta}_{\alpha_{1}}=\left(\eta_{\mu}^{A}, \eta\right),  \tag{52}\\
& \hat{\Phi}_{\alpha_{0}}^{*}=\left(h_{A}^{* \mu \nu}, V^{* \mu}\right), \quad \hat{\eta}^{* \alpha_{1}}=\left(\eta_{A}^{* \mu}, \eta^{*}\right), \tag{53}
\end{align*}
$$

whose degrees are the same like in the case of a single Pauli-Fierz field. The BRST differential decomposes exactly like in (5) and its components act on the BRST generators via the relations

$$
\begin{align*}
\delta h_{A}^{* \mu \nu} & =2 H_{A}^{\mu \nu}, \quad \delta V^{* \mu}=-\partial_{\nu} F^{\nu \mu},  \tag{54}\\
\delta \eta_{A}^{* \mu} & =-2 \partial_{\nu} h_{A}^{* \nu \mu}, \quad \delta \eta^{*}=-\partial_{\mu} V^{* \mu},  \tag{55}\\
\delta \hat{\Phi}^{\alpha_{0}} & =0, \quad \delta \hat{\eta}_{\alpha_{1}}=0,  \tag{56}\\
\gamma \hat{\Phi}_{\alpha_{0}}^{*} & =0, \quad \gamma \hat{\eta}^{* \alpha_{1}}=0,  \tag{57}\\
\gamma h_{\mu \nu}^{A} & =\partial_{(\mu} \eta_{\nu)}^{A}, \quad \gamma V_{\mu}=\partial_{\mu} \eta,  \tag{58}\\
\gamma \eta_{\mu}^{A} & =0, \quad \gamma \eta=0, \tag{59}
\end{align*}
$$

where $H_{A}^{\mu \nu}=K_{A}^{\mu \nu}-\frac{1}{2} \sigma^{\mu \nu} K_{A}$ is the linearized Einstein tensor of the Pauli-Fierz field $h_{A}^{\mu \nu}$. The solution to the master equation for this free model takes the simple form

$$
\begin{equation*}
\bar{S}^{\prime}=S_{0}^{\mathrm{L}}\left[h_{\mu \nu}^{A}, V_{\mu}\right]+\int d^{D} x\left(h_{A}^{* \mu \nu} \partial_{(\mu} \eta_{\nu)}^{A}+V^{* \mu} \partial_{\mu} \eta\right) . \tag{60}
\end{equation*}
$$

### 4.1.2 First-order deformation

Acting like in the one graviton case, we can write the first-order deformation of the solution to the master for a single vector field and a collection of Pauli-Fierz fields like

$$
\begin{equation*}
\hat{S}_{1}=\hat{S}_{1}^{(\mathrm{PF})}+\hat{S}_{1}^{\text {(int })} \tag{61}
\end{equation*}
$$

where

$$
\begin{align*}
\hat{S}_{1}^{\text {(PF) }}= & \int d^{D} x\left\{\frac{1}{2} f_{B C}^{A} \eta_{A}^{* \mu} \eta^{B \nu} \partial_{[\mu} \eta_{\nu]}^{C}+f_{B C}^{A} h_{A}^{* \mu \rho}\left[\left(\partial_{\rho} \eta^{B \nu}\right) h_{\mu \nu}^{C}\right.\right. \\
& \left.\left.-\eta^{B \nu} \partial_{[\mu} h_{\nu] \rho}^{C}\right]+f_{A B C} \hat{a}_{0}^{(\mathrm{cubic}) A B C}-2 \Lambda_{A} h^{A}\right\},  \tag{62}\\
\hat{S}_{1}^{\text {(int })}= & \int d^{D} x\left\{y_{2 A}\left[h^{* A} \eta+(D-2)\left(-V^{* \lambda} \eta_{\lambda}^{A}+V^{\lambda} \partial_{[\mu} h_{\lambda]}^{A \mu}\right)\right]\right. \\
& +y_{3}^{A} \delta_{3}^{D} \varepsilon_{\mu \nu \rho}\left(V^{* \mu} \partial^{[\nu} \eta_{A}^{\rho]}+F^{\lambda \mu} \partial^{[\nu} h_{A \lambda}^{\rho]}\right)+p_{A}\left[\eta^{*} \eta_{\mu}^{A} \partial^{\mu} \eta\right. \\
& -\frac{1}{2} V^{* \mu}\left(V^{\nu} \partial_{[\mu} \eta_{\nu]}^{A}+2\left(\partial_{\nu} V_{\mu}\right) \eta^{A \nu}-h_{\mu \nu}^{A} \partial^{\nu} \eta\right) \\
& \left.+\frac{1}{8} F^{\mu \nu}\left(2 \partial_{[\mu}\left(h_{\nu] \rho}^{A} V^{\rho}\right)+F_{\mu \nu} h^{A}-4 F_{\mu \rho} h_{\nu}^{A \rho}\right)\right] \\
& \left.+q_{1} \delta_{3}^{D} \varepsilon^{\mu \nu \lambda} V_{\mu} F_{\nu \lambda}+q_{2} \delta_{5}^{D} \varepsilon^{\mu \nu \lambda \alpha \beta} V_{\mu} F_{\nu \lambda} F_{\alpha \beta}\right\} . \tag{63}
\end{align*}
$$

We remark that the first-order deformation is parameterized by seven types of real, constant coefficients, namely $f_{B C}^{A}, \Lambda_{A}, y_{2 A}, y_{3}^{A} \delta_{3}^{D}, p_{A}, q_{1} \delta_{3}^{D}$, and $q_{2} \delta_{5}^{D}$, with $f_{A B C}$ defined as

$$
\begin{equation*}
f_{A B C}=k_{A D} f_{B C}^{D} \equiv \delta_{A D} f_{B C}^{D} . \tag{64}
\end{equation*}
$$

Moreover, the construction of $\hat{S}_{1}^{(\mathrm{PF})}$ requires that the constants $f_{B C}^{A}$ and $f_{A B C}$ must satisfy the relations

$$
\begin{align*}
f_{B C}^{A} & =f_{C B}^{A}  \tag{65}\\
f_{A B C} & =\frac{1}{3}\left(f_{A B C}+f_{C A B}+f_{B C A}\right) \equiv \frac{1}{3} f_{(A B C)} \tag{66}
\end{align*}
$$

### 4.1.3 Consistency of the first-order deformation

Next, we investigate the consistency of the first-order deformation, expressed by equation (22), with $S_{1,2}$ replaced by $\hat{S}_{1,2}$

$$
\begin{equation*}
\left(\hat{S}_{1}, \hat{S}_{1}\right)+2 s \hat{S}_{2}=0 \tag{67}
\end{equation*}
$$

On the one hand, the equation (67) restricts the coefficients $f_{A B}^{C}$ to satisfy the supplementary conditions

$$
\begin{equation*}
f_{A[B}^{D} f_{C] D}^{E}=0 . \tag{68}
\end{equation*}
$$

Combining (65), (66), and (68), we conclude that the coefficients $f_{A B}^{C}$ define the structure constants of a real, commutative, symmetric, and associative (finite-dimensional) algebra. Such an algebra has a trivial structure: it is a direct sum of one-dimensional ideals. Therefore, $f_{A B}^{C}=0$ whenever two indices are different

$$
\begin{equation*}
f_{A B}^{C}=0, \quad \text { if } \quad(A \neq B \quad \text { or } \quad B \neq C \quad \text { or } \quad C \neq A) \tag{69}
\end{equation*}
$$

For notational simplicity, we denote $f_{A B C}$ for $A=B=C$ by

$$
\begin{equation*}
f_{A A A} \equiv f_{A} \quad \text { without summation over } A \tag{70}
\end{equation*}
$$

On the other hand, the equation (67) leads to the following relations

$$
\begin{align*}
& p_{A} p_{B}=0,  \tag{71}\\
& \text { for all } \quad A \neq B,  \tag{72}\\
&\left(p_{A} y_{3 B}+p_{B} y_{3 A}\right) \delta_{3}^{D}=0,  \tag{73}\\
& p_{A} y_{2 B}+p_{B} y_{2 A}=0,  \tag{74}\\
& p_{A}\left(f_{A}+p_{A}\right)=0,
\end{align*} \quad \text { for all } \quad A \neq B, \quad A \neq B,
$$

$$
\begin{align*}
\left(f_{A}+2 p_{A}\right) y_{3 A} \delta_{3}^{D} & =0, \quad \text { without summation over } A,  \tag{75}\\
\left(f_{A}+2 p_{A}\right) y_{2 A} & =0, \quad \text { without summation over } A . \tag{76}
\end{align*}
$$

Equations (71)-(76) possess two types of solutions. In case I we have the solution

$$
\begin{equation*}
p_{1}=-f_{1} \neq 0, \quad\left(p_{B}\right)_{B=\overline{2, n}}=0, \quad\left(y_{3 A} \delta_{3}^{D}\right)_{A=\overline{1, n}}=0=\left(y_{2 A}\right)_{A=\overline{1, n}}, \tag{77}
\end{equation*}
$$

while in case II the solution reads as

$$
\begin{equation*}
\left(p_{A}\right)_{A=\overline{1, n}}=0, \quad\left(f_{\bar{A}}\right)_{\bar{A}=\overline{1, m}}=0, \quad\left(y_{3 A^{\prime}} \delta_{3}^{D}\right)_{A^{\prime}=\overline{m+1, n}}=0=\left(y_{2 A^{\prime}}\right)_{A^{\prime}=\overline{m+1, n}} . \tag{78}
\end{equation*}
$$

The solution (78) contains two limit situations: $m=n$ and $m=0$.
In general, the consistency of the deformations of orders three, four, and higher may impose new restrictions upon the constants that parameterize the first-order deformation.

### 4.2 Main cases. Coupled theories

### 4.2.1 Case I: no-go results in General Relativity

In case I, the consistency procedure imposes no new constraints on the parameterizing constants. In consequence, the coupled model is maximally parameterized by $\left(f_{A}\right)_{A=\overline{1, n}}, p_{1}=-f_{1} \neq 0,\left(\Lambda_{A}\right)_{A=\overline{1, n}}$, $q_{1} \delta_{3}^{D}$, and $q_{2} \delta_{5}^{D}$. Of course, it is possible that some of $f_{B}($ for $B \neq 1), \Lambda_{A}, q_{1}$, or $q_{2}$ vanish. Accordingly, in case I we obtain the Lagrangian action (for $f_{1}=1=-p_{1}$ )

$$
\begin{align*}
& \quad \hat{S}^{\mathrm{L}(\mathrm{I})}\left[h_{\mu \nu}^{A}, V_{\mu}\right]=\int d^{D} x\left[\frac{2}{k^{2}} \sqrt{-g^{1}}\left(R^{1}-2 k^{2} \Lambda_{1}\right)\right. \\
& \quad-\frac{1}{4} \sqrt{-g^{1}} g^{1 \mu \nu} g^{1 \rho \lambda} \bar{F}_{\mu \rho}^{1} \bar{F}_{\nu \lambda}^{1}+k\left(q_{1} \delta_{3}^{D} \varepsilon^{1 \mu_{1} \mu_{2} \mu_{3}} \bar{V}_{\mu_{1}}^{1} \bar{F}_{\mu_{2} \mu_{3}}^{1}\right. \\
& \left.\left.+q_{2} \delta_{5}^{D} \varepsilon^{1 \mu_{1} \mu_{2} \mu_{3} \mu_{4} \mu_{5}} \bar{V}_{\mu_{1}}^{1} \bar{F}_{\mu_{2} \mu_{3}}^{1} \bar{F}_{\mu_{4} \mu_{5}}^{1}\right)\right] \\
& \quad+\sum_{B=2}^{n}\left[\int d^{D} x \frac{2}{k_{B}^{2}} \sqrt{-g^{B}}\left(R^{B}-2 k k_{B} \Lambda_{B}\right)\right] \\
& \equiv  \tag{79}\\
& \hat{S}^{\mathrm{L}(\mathrm{I})}\left[g_{\mu \nu}^{1}, \bar{V}_{\mu}^{1}\right]+\sum_{B=2}^{n} \hat{S}^{\mathrm{L}(\mathrm{E}-\mathrm{H})}\left[g_{\mu \nu}^{B}\right],
\end{align*}
$$

where $\bar{V}_{\mu}^{1}$ and $\bar{F}_{\mu \nu}^{1}$ are 'curved' with the vielbein fields from the first graviton sector

$$
\begin{align*}
\bar{V}_{\mu}^{1} & =e_{\mu}^{1 a} V_{a}, \quad \bar{F}_{\mu \nu}^{1}=\partial_{[\mu}\left(e_{\nu]}^{11} V_{a}\right),  \tag{80}\\
\varepsilon^{1 \mu_{1} \mu_{2} \ldots \mu_{D}} & =\sqrt{-g^{1}} e_{a_{1}}^{1 \mu_{1}} \cdots e_{a_{D}}^{1 \mu_{D}} \varepsilon^{a_{1} \ldots a_{D}} . \tag{81}
\end{align*}
$$

The notations $R^{A}$ and $g^{A}(A=\overline{1, n})$ denote the full scalar curvature and the determinant of the metric tensor $g_{\mu \nu}^{A}=\sigma_{\mu \nu}+k_{A} h_{\mu \nu}^{A}$ (without summation over $A$ ) from the $A$-th graviton sector respectively, while $k_{B}=k f_{B}, B=\overline{2, n}$. From (79), we observe that the vector field gets coupled to single graviton ( $A=1$ ) according to the standard coupling from General Relativity, while each of the other gravitons ( $B=\overline{2, n}$ ) interacts only with itself according to an Einstein-Hilbert action (or possibly a Pauli-Fierz action if $f_{B}=0$ ) with a cosmological term.

The final conclusion is that in the first case there is no cross-interaction among different gravitons to all orders in the coupling constant.

### 4.2.2 Case II: no-go results for the new couplings in $D=3$

In case II, the consistency procedure imposes new constraints on the parameterizing constants, such that we obtain two different situations.

Subcase II. 1 In subcase II. 1 the consistency of the deformed solution to the master equation requires the conditions

$$
\begin{align*}
\left(p_{A}\right)_{A=\overline{1, n}} & =0=\left(y_{2 A}\right)_{A=\overline{1, n}}, \quad\left(f_{\bar{A}}\right)_{\bar{A}=\overline{1, m}}=0  \tag{82}\\
\left(y_{3 A^{\prime}} \delta_{3}^{D}\right)_{A^{\prime}=\overline{m+1, n}} & =0, \quad q_{1} \delta_{3}^{D}=0=q_{2} \delta_{5}^{D} \tag{83}
\end{align*}
$$

The full deformed Lagrangian action for this subcase is given by

$$
\begin{equation*}
\hat{S}^{\mathrm{L}(\mathrm{II} .1)}=\sum_{A^{\prime}=m+1}^{n} \hat{S}^{\mathrm{L}(\mathrm{E}-\mathrm{H})}\left[g_{\mu \nu}^{A^{\prime}}\right]+\hat{S}^{\mathrm{L}(\text { special })} \tag{84}
\end{equation*}
$$

where

$$
\begin{align*}
& \hat{S}^{\mathrm{L}(\text { special })}=\sum_{\bar{A}=1}^{m}\left\{\int d^{D} x\left[\mathcal{L}_{0}^{(\mathrm{PF})}\left(h_{\mu \nu}^{\bar{A}}, \partial_{\lambda} h_{\mu \nu}^{\bar{A}}\right)-2 k \Lambda_{\bar{A}} h^{\bar{A}}\right]\right\} \\
& +\int d^{D} x\left\{-\frac{1}{4} F_{\mu \nu} F^{\mu \nu}+k \sum_{\bar{A}=1}^{m} y_{3}^{\bar{A}} \delta_{3}^{D} \varepsilon^{\mu \nu \rho} F_{\lambda \mu} \partial_{[\nu} h_{\rho]}^{\bar{A} \lambda}\right. \\
& \left.+k^{2} \sum_{\bar{A}, \bar{B}=1}^{m}\left[y_{3}^{\bar{A}} y_{3}^{\bar{B}} \delta_{3}^{D}\left(\partial_{[\nu} h_{\rho] \lambda}^{\bar{A}}\right) \partial_{\left[\nu^{\prime}\right.} h_{\left.\rho^{\prime}\right]}^{\bar{B}} \lambda \sigma^{\nu \nu^{\prime}} \sigma^{\rho \rho^{\prime}}\right]\right\} . \tag{85}
\end{align*}
$$

Each $\hat{S}^{\mathrm{L}(\mathrm{E}-\mathrm{H})}\left[g_{\mu \nu}^{A^{\prime}}\right]$ represents a copy of the full Einstein-Hilbert action with a cosmological constant associated with the graviton field $h_{\mu \nu}^{A^{\prime}}\left(A^{\prime}=\overline{m+1, n}\right)$, so they cannot produce couplings among different gravitons. Let us analyze in more detail the second part. It stops at order two in the coupling constant and in $D=3$ spacetime dimensions seems to mix different spin-two fields via the terms from the last (double) sum in the right-hand side of (85) with $\bar{A} \neq \bar{B}$.

In order to focus in more detail on (85) we take the limit situation $m=n$ (so $\bar{A} \rightarrow A$ ) in the conditions (82)-(83) and work in $D=3$, such that the entire deformed action, $\hat{S}^{\mathrm{L}(I I .1)}$, reduces to (85), i.e.

$$
\begin{align*}
\hat{S}^{\mathrm{L}(\mathrm{II.1)}}\left[h_{\mu \nu}^{A}, V^{\mu}\right]= & \int d^{3} x\left\{-\frac{1}{4} F_{\mu \nu} F^{\mu \nu}+\sum_{A=1}^{n}\left[\mathcal{L}_{0}^{(\mathrm{PF})}\left(h_{\mu \nu}^{A}, \partial_{\lambda} h_{\mu \nu}^{A}\right)\right.\right. \\
& \left.-2 k \Lambda_{A} h^{A}-k y_{3}^{A} \varepsilon^{\mu \nu \rho} F_{\mu \nu} \partial_{[\theta} h_{\rho]}^{A \theta}\right] \\
& \left.+2 k^{2} \sum_{A, B=1}^{n}\left[y_{3}^{A} y_{3}^{B}\left(\partial_{[\mu} h_{\rho]}^{A \mu}\right) \partial_{[\nu} h_{\lambda]}^{B} \sigma^{\rho \lambda}\right]\right\} \tag{86}
\end{align*}
$$

Unfortunately, action (86) does not describe in fact cross-couplings between different spin-two fields. In order to make this observation clear, let us denote by $Y$ the matrix of elements $y_{3}^{A} y_{3}^{B}$. It is simple to see that the rank of $Y$ is equal to one. By an orthogonal transformation $M$ we can always find a matrix $\hat{Y}$ of the form

$$
\begin{equation*}
\hat{Y}=M^{T} Y M \tag{87}
\end{equation*}
$$

with $M^{T}$ the transposed of $M$, such that

$$
\begin{equation*}
\hat{Y}^{11}=\sum_{A=1}^{n}\left(y_{3}^{A}\right)^{2} \equiv \lambda^{2}, \quad \hat{Y}^{1 A^{\prime}}=\hat{Y}^{B^{\prime} 1}=\hat{Y}^{A^{\prime} B^{\prime}}=0, \quad A^{\prime}, B^{\prime}=\overline{2, n} \tag{88}
\end{equation*}
$$

If we make the notation

$$
\begin{equation*}
\hat{y}^{A}=M^{A C} y_{3}^{C} \tag{89}
\end{equation*}
$$

then relation (88) implies

$$
\begin{equation*}
\hat{y}^{A}=\lambda \delta_{1}^{A} \tag{90}
\end{equation*}
$$

Now, we make the field redefinition

$$
\begin{equation*}
h_{\mu \nu}^{A}=M^{A C} \hat{h}_{\mu \nu}^{C} \tag{91}
\end{equation*}
$$

with $M^{A C}$ the elements of $M$. This transformation of the spin-two fields leaves $\sum_{A=1}^{n} \mathcal{L}_{0}^{(\mathrm{PF})}\left(h_{\mu \nu}^{A}, \partial_{\lambda} h_{\mu \nu}^{A}\right)$ invariant and, moreover, based on the above results, we obtain

$$
\begin{gather*}
\sum_{A, B=1}^{n}\left[y_{3}^{A} y_{3}^{B}\left(\partial_{[\mu} h_{\rho]}^{A \mu}\right) \partial_{[\nu} h_{\lambda]}^{B} \nu^{\rho \lambda}\right]=\lambda^{2}\left(\partial_{[\mu} \hat{h}_{\rho]}^{1 \mu}\right) \partial_{[\nu} \hat{h}_{\lambda]}^{1 \nu} \sigma^{\rho \lambda}  \tag{92}\\
\sum_{A=1}^{n}\left(y_{3}^{A} \varepsilon^{\mu \nu \rho} F_{\mu \nu} \partial_{[\theta} h_{\rho]}^{A \theta}\right)=\lambda \varepsilon^{\mu \nu \rho} F_{\mu \nu} \partial_{[\theta} \hat{h}_{\rho]}^{1} \theta \tag{93}
\end{gather*}
$$

such that (??) becomes

$$
\begin{equation*}
\hat{S}^{\mathrm{L}(\mathrm{II} .1)}\left[\hat{h}_{\mu \nu}^{A}, V^{\mu}\right]=\int d^{3} x\left[\sum_{A=1}^{n}\left(\mathcal{L}_{0}^{(\mathrm{PF})}\left(\hat{h}_{\mu \nu}^{A}, \partial_{\lambda} \hat{h}_{\mu \nu}^{A}\right)-2 k \hat{\Lambda}_{A} \hat{h}^{A}\right)-\frac{1}{4} \hat{F}_{\mu \nu}^{\prime} \hat{F}^{\prime \mu \nu}\right] \tag{94}
\end{equation*}
$$

where

$$
\begin{equation*}
\hat{\Lambda}_{A}=\Lambda_{B} M^{B A}, \quad \hat{F}^{\prime \mu \nu}=F^{\mu \nu}+2 k \lambda \varepsilon^{\mu \nu \rho} \partial_{[\theta} \hat{h}_{\rho]}^{1} \theta \tag{95}
\end{equation*}
$$

We observe that action (94) decouples into action (47) for the first spin-two field $(A=1)$ and a sum of Pauli-Fierz actions with cosmological terms for the remaining $(n-1)$ spin-two fields. In conclusion, we cannot couple different spin-two fields even outside the framework of General Relativity.

Subcase II. 2 Here, we have three subsubcases. The first subsubcase is given by the conditions

$$
\begin{align*}
\left(p_{A}\right)_{A=\overline{1, n}} & =0, \quad\left(f_{\bar{A}}\right)_{\bar{A}=\overline{1, m}}=0, \quad\left(y_{3 A^{\prime}} \delta_{3}^{D}\right)_{A^{\prime}=\overline{m+1, n}}=0  \tag{96}\\
\left(y_{2 A}\right)_{A=\overline{1, n}} & =0, \quad D \neq 3 \tag{97}
\end{align*}
$$

In this context, we obtain the deformed Lagrangian action

$$
\begin{align*}
\hat{S}^{\mathrm{L}(\mathrm{II} .2)}= & \sum_{\bar{A}=1}^{m}\left\{\int d^{D} x\left[\mathcal{L}_{0}^{(\mathrm{PF})}\left(h_{\mu \nu}^{\bar{A}}, \partial_{\lambda} h_{\mu \nu}^{\bar{A}}\right)-2 k \Lambda_{\bar{A}} h^{\bar{A}}\right]\right\} \\
& +\sum_{A^{\prime}=m+1}^{n} \hat{S}^{\mathrm{L}(\mathrm{E}-\mathrm{H})}\left[g_{\mu \nu}^{A^{\prime}}\right]-\frac{1}{4} F_{\mu \nu} F^{\mu \nu} \\
& +k q_{2} \delta_{5}^{D} \int d^{D} x \varepsilon^{\mu \nu \lambda \alpha \beta} V_{\mu} F_{\nu \lambda} F_{\alpha \beta} \tag{98}
\end{align*}
$$

so, there are no couplings among different gravitons.
The second subsubcase is described by the relations

$$
\begin{align*}
\left(p_{A}\right)_{A=\overline{1, n}} & =0, \quad\left(f_{\bar{A}}\right)_{\bar{A}=\overline{1, m}}=0, \quad\left(y_{3 A^{\prime}} \delta_{3}^{D}\right)_{A^{\prime}=\overline{m+1, n}}=0  \tag{99}\\
\left(y_{2 A}\right)_{A=\overline{1, n}} & =0, \quad q_{1}=0, \quad D=3 \tag{100}
\end{align*}
$$

In this situation we re-obtain the case from the previous section, described by formula (84), where we have shown that there are no cross-couplings between different gravitons.

The third subsubcase corresponds to the formulas

$$
\begin{align*}
\left(p_{A}\right)_{A=\overline{1, n}} & =0, \quad\left(f_{\bar{A}}\right)_{\bar{A}=\overline{1, m}}=0, \quad\left(y_{3 A}\right)_{A=\overline{1, n}}=0,  \tag{101}\\
\left(y_{2 A}\right)_{A=\overline{1, n}} & =0, \quad D=3 . \tag{102}
\end{align*}
$$

In this subsubcase, again no cross-couplings are permitted due to the fact that the resulting Lagrangian is like in (98) up to replacing the density $k q_{2} \delta_{5}^{D} \varepsilon^{\mu \nu \lambda \alpha \beta} V_{\mu} F_{\nu \lambda} F_{\alpha \beta}$ with the standard Abelian ChernSimons term $k q_{1} \varepsilon^{\mu \nu \lambda} V_{\mu} F_{\nu \lambda}$.

## 5 Generalization to an arbitrary p-form

The results obtained so far in the presence of a massless vector field can be generalized to the case of deformations for one or several gravitons and an arbitrary $p$-form gauge field.

In the case of a single graviton the starting point is the sum between the Pauli-Fierz action and the Lagrangian action of an Abelian $p$-form with $p>1$

$$
\begin{equation*}
S_{0}^{\mathrm{L}}\left[h_{\mu \nu}, V_{\mu_{1} \ldots \mu_{p}}\right]=\int d^{D} x\left(\mathcal{L}_{0}^{(\mathrm{PF})}-\frac{1}{2 \cdot(p+1)!} F_{\mu_{1} \ldots \mu_{p+1}} F^{\mu_{1} \ldots \mu_{p+1}}\right) \tag{103}
\end{equation*}
$$

in $D \geq p+1$ spacetime dimensions, with $F_{\mu_{1} \ldots \mu_{p+1}}$ the Abelian field strength of the $p$-form gauge field $V_{\mu_{1} \ldots \mu_{p}}$

$$
\begin{equation*}
F_{\mu_{1} \ldots \mu_{p+1}}=\partial_{\left[\mu_{1}\right.} V_{\left.\mu_{2} \ldots \mu_{p+1}\right]} . \tag{104}
\end{equation*}
$$

This action is known to be invariant under the gauge transformations

$$
\begin{equation*}
\delta_{\epsilon} h_{\mu \nu}=\partial_{(\mu} \epsilon_{\nu)}, \quad \delta_{\epsilon} V_{\mu_{1} \ldots \mu_{p}}=\partial_{\left[\mu_{1}\right.} \epsilon_{\mu_{2} \ldots \mu_{p]}}^{(p)} \tag{105}
\end{equation*}
$$

Unlike the Maxwell field $(p=1)$, the gauge transformations of the $p$-form for $p>1$ are off-shell reducible of order $(p-1)$. This property has strong implications at the level of the BRST complex and of the BRST cohomology in the form sector: a whole tower of ghosts of ghosts and of antifields will be required in order to incorporate the reducibility, only the ghost of maximum pure ghost number, $p$, will enter $H(\gamma)$, and the local characteristic cohomology will be richer in the sense that (30) and (32) become

$$
\begin{equation*}
H_{J}(\delta \mid d)=0=H_{J}^{\mathrm{inv}}(\delta \mid d), \quad J>p+1 \tag{106}
\end{equation*}
$$

In spite of these new cohomological ingredients, which complicate the analysis of deformations, the previous results can still be generalized.

Thus, two complementary cases are revealed. One describes the standard graviton-p-form interactions from General Relativity and leads to a Lagrangian action similar to (43) up to replacing $(1 / 4) g^{\mu \nu} g^{\rho \lambda} \bar{F}_{\mu \rho} \bar{F}_{\nu \lambda}$ with the expression $(2 \cdot(p+1)!)^{-1} g^{\mu_{1} \nu_{1}} \cdots g^{\mu_{p+1} \nu_{p+1}} \bar{F}_{\mu_{1} \ldots \mu_{p+1}} \bar{F}_{\nu_{1} \ldots \nu_{p+1}}$ and, if $p$ is odd, also the terms containing $\delta_{3}^{D} \varepsilon^{\mu_{1} \mu_{2} \mu_{3}}$ and $\delta_{5}^{D} \varepsilon^{\mu_{1} \mu_{2} \mu_{3} \mu_{4} \mu_{5}}$ with some densities involving $\delta_{2 p+1}^{D} \varepsilon^{\mu_{1} \ldots \mu_{2 p+1}}$ and $\delta_{3 p+2}^{D} \varepsilon^{\mu_{1} \ldots \mu_{3 p+2}}$ respectively (if $p$ is even, the terms proportional with either $q_{1}$ or $q_{2}$ must be suppressed). The other case emphasizes that it is possible to construct some new deformations in $D=p+2$, describing a spin two-field coupled to a p-form and having (103) and (105) as a free limit, which are consistent to all orders in the coupling constant and are not subject to the rules of General Relativity. Their source is a generalization of the terms proportional with $y_{3}$ from the first-order deformation (34)

$$
\begin{equation*}
S_{1}^{(\mathrm{int})}\left(y_{3}\right)=y_{3} \varepsilon_{\mu_{1} \ldots \mu_{p} \nu \rho} \int d^{p+2} x\left(V^{* \mu_{1} \ldots \mu_{p}} \partial^{[\nu} \eta^{\rho]}+\frac{1}{p!} F^{\lambda \mu_{1} \ldots \mu_{p}} \partial^{[\nu} h_{\lambda}^{\rho]}\right) . \tag{107}
\end{equation*}
$$

Performing the necessary computations, we find the Lagrangian action

$$
\begin{align*}
S^{\mathrm{L}}\left[h_{\mu \nu}, V_{\mu_{1} \ldots \mu_{p}}\right]= & \int d^{p+2} x\left(\mathcal{L}_{0}^{(\mathrm{PF})}-2 k \Lambda h\right. \\
& \left.-\frac{1}{2 \cdot(p+1)!} F_{\mu_{1} \ldots \mu_{p+1}}^{\prime} F^{\prime \mu_{1} \ldots \mu_{p+1}}\right) \tag{108}
\end{align*}
$$

where the field strength of the $p$-form is deformed as

$$
\begin{equation*}
F_{\mu_{1} \ldots \mu_{p+1}}^{\prime}=F_{\mu_{1} \ldots \mu_{p+1}}+2(-)^{p+1} k y_{3} \varepsilon_{\mu_{1} \ldots \mu_{p+1} \rho} \partial^{[\theta} h_{\theta}^{\rho]} \tag{109}
\end{equation*}
$$

This action is fully invariant under the original Pauli-Fierz gauge transformations and

$$
\begin{equation*}
\bar{\delta}_{\epsilon} V_{\mu_{1} \ldots \mu_{p}}=\partial_{\left[\mu_{1}\right.} \epsilon_{\mu_{2} \ldots \mu_{p]}}^{(p)}+k y_{3} \varepsilon_{\mu_{1} \ldots \mu_{p} \nu \rho} \partial^{[\nu} \epsilon^{\rho]} \tag{110}
\end{equation*}
$$

The gauge algebra remains Abelian and the reducibility of (110) is not affected by these couplings: the associated functions and relations are the initial ones.

Regarding a collection of spin-two fields and a $p$-form, we find that there are no cross-couplings among different spin-two fields intermediated by a $p$-form gauge field: the $p$-form couples to a single spin-two field.

## 6 Conclusion

Under the hypotheses of space-time locality, smoothness in the coupling constant, (background) Lorentz covariance, Poincaré invariance, and preservation of the number of derivatives on each field (plus positivity of the metric in the internal space in the case of a collection of spin-two fields), we proved that there are no indirect consistent cross-interactions among different gravitons in the presence of a massless $p$-form.

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