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Complete dynamics in the extended phase-space

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Abstract

The paper points out the importance of the concept of symmetry in two (apparently) different domains: QFT and Nonlinear dynamics. Two fundamental types of symmetries will be considered: the point-like and the gauge symmetries. The BRST symmetry and its extensions will be implemented as starting point for some nonlinear mechanical models. We will investigate the influences that the ghost fields from the BRST Hamiltonian formalism could have on the dynamics of the gauge fields when the ghosts are seen as real fields. The evolutionary equations for the ghosts are written down and these equations are coupled with those corresponding to the real fields. The Yang-Mills field is considered as an example and some features for a $2D$ nonlinear dynamical model arising from this approach are compared with the similar results obtained for the free Yang-Mills mechanical model.

Key words: extended BRST symmetry, Yang-Mills field, mechanical models

1 Introduction

Our lecture will concentrate on a very important concept of Physics: the symmetry. We will look for the meaning of this concept in two apparently different chapter of Theoretical Physics: the Quantum Field Theory and the Nonlinear Dynamics. More precisely we will investigate and we will try to connect the BRST symmetry, a global symmetry which embed all the local symmetries of a gauge field theory, and the Lie type symmetries, important for describing the evolution of some nonlinear dynamical systems with a finite number of degree of freedom (mechanical systems).

The BRST symmetry [1] offers an appropriate frame in which a covariant description of the gauge theories becomes possible. Many interesting models such as the Yang-Mills theories, have been successfully investigated by using this approach. Its starting point is given by the original action of the considered model, action which in the BRST approach has to be extended with ghost type fields in order to generate correct equations of motion. Usually, the equations for the real fields only are taken into account. Ghosts are considered as having no physical significance, in the sense that they disappear in the asymptotic states.

One of the ideas of this paper is to present what is happening when both the real and the ghost variables are seen as real generators of the dynamics, and the whole system of equations of motion is taken into consideration. The $sp(3)$ Hamiltonian description of the Yang-Mills theory is presented as a concrete example. After writing down all the equations of motion, both for real fields and for ghosts, the introduction of some specific expressions of all these fields in terms of color factors, generates interesting examples of nonlinear mechanical models [3]. Many reasons determined us to choose the $sp(3)$ BRST approach. It had be noticed that, in the most general context, the BRST symmetry can be represented as a sum of anticommuting differential operators [4]:

$$s = s_1 + s_2 + ...s_n;\ s_a s_b + s_b s_a = 0 \tag{1}$$

The standard symmetry is given by s_1 only. The $sp(2)$ formalism generates the first two pieces, s_1 and s_2, of the generalized $sp(n)$ representation. The $sp(3)$ is the next step ahead and it offers many possibilities for the gauge fixing procedure. On the other part, because in the $sp(3)$ case the whole spectrum of the variables is larger, an interesting nonlinear mechanical model is generated when the

Two main results are reported here: (i) the equations of evolution for the ghosts allow to eliminate all the ghost type variables , the terms from gauge fixed action which contain these variables are automatically eliminated; (ii) eliminated, slight traces of regularity have been pointed out by the emergence of periodical orbits.

The paper is structured as follows: after the preliminary considerations, the second section make up a brief introduction to the two types of symmetries we are dealing with: the Lie symmetries on one hand and the extended $sp(3)$ BRST symmetry on the other. In the third section, we apply these general results to the case of the Yang-Mills theory, seen as a field theory. The next section of the paper makes the transition to the extended mechanical Yang-Mills model, providing the whole set of equations that has to be considered in order to obtain a coherent description. These equations are then studied for a particular 4-dimensional case and concrete results concerning the dynamics of the system are presented. Some concluding remarks will end the paper.

2 From the Lie to the BRST symmetries

2.1 Lie symmetries of the mechanical models

A point-like transformation in the $(\overrightarrow{x}, t)$ space-time may be defined through an infinitesimal parameter ε by:

$$\begin{array}{rcl} & & t' - t + \delta t,\ \delta t - \varepsilon\varphi(x,t) \\ x_i'(t) & = & x_i(t) + \delta x_i,\ \delta x_i = \varepsilon\xi_i(x,t);\ i = 1, ..., n \end{array} \tag{2}$$

The variation of an arbitrary analytical function $u(x,t),\ \delta u = u(x', t\prime) - u(x,t)$:

$$\delta u = \frac{\partial u}{\partial t}\delta t + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\delta x_i = \varepsilon U u(x,t) \tag{3}$$

The operator U denotes the generator of the infinitesimal point-like transformation and is called *Lie operator*. Its concrete form is:

$$U = \varphi\frac{\partial}{\partial t} + \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,t)\frac{\partial}{\partial u^\alpha} \tag{4}$$

The extension of the n-th order [5]:

$$U^{(n)} = U + \sum_{\alpha=1}^{q}\sum_{J} \phi_\alpha^J(x, u^{(n)})\frac{\partial}{\partial u_J^\alpha} \tag{5}$$

where:

$$u_J^\alpha = \frac{\partial^J u^\alpha}{\partial x^{j_1}\partial x^{j_2}...\partial x^{j_m}},\ J = j_1 + j_2 + ...j_m \tag{6}$$

Let us consider a nonlinear dynamical system described by a n-th order partial derivative system of equations:

$$\Delta_\nu(x, u^{(n)}[x]) = 0\ ;\ x = \{x^i, i = 1, ..., p\}; u = \{u^\alpha, \alpha = 1, ...q\} \tag{7}$$

The operator (4) leaves invariant this system iff [6]:

$$U^{(n)}[\Delta_\nu] = 0 \tag{8}$$

An interesting extension of the concept of Lie symmetry is those of the nonclassical symmetries of the system [7]. The basic idea of the non-classical method is to add to (8) the requirement of *invariance surface condition*. It has the form:

$$Q^\alpha(x, u^{(1)}) = \phi^\alpha(x,u) - \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial u^\alpha}{\partial x_i}, \alpha - 1, ..., q \tag{9}$$

The set $Q \equiv \{Q^1, Q^2, ...Q^q\}$ is known as the characteristic of the symmetry operator and acts as a system of constraints. The number of determining equations is smaller. Consequently, the set of solutions is larger than the one for the classical method.

The study of the point-like (Lie) symmetries of a dynamical systems is very important especially for the cases when do not exist concrete solutions of the equations of motion. The symmetries allows to obtain invariant quantities and, eventually, conservation laws.

2.2 The standard BRST symmetry

The standard BRST symmetry designates a global symmetry which, as Becchi, Rouet, Stora and later Tyutin showed, can be attached to any physical system and can embed any other local symmetry the system could have. Denominations like "standard" symmetry have the role of distinguishing between the symmetry as it was pointed out at the beginning and later extensions. In the next section, we shall discuss one of these extensions, namely the sp(3) symmetry. It will be in fact the working frame of our approach.

The special importance of the BRST symmetry is connected with the possibility of achieving a coherent quantum description of the singular systems (gauge theories or constrained dynamical systems). The idea issued from papers of Fadeev and Popov [8] who have used for the first time "non-physical" quantities in the Lagrangian description of the Yang-Mills fields. Their approach was further developed by t'Hooft [9], Feynman [10] and de Witt [11]. These quantities were called ghost fields and they do not appear in the asymptotic states of quantum field theories. This is way their dynamics is not usually taken into account. As already mentioned, we shall consider their dynamics, too.

The starting point of the BRST theory is the action S_0 which describe the singular system. It has to be replaced by an extended action, $S = S_0 + ...$, which will contain the local symmetries of S_0 but will spoil it from its non-physical degrees of freedom and will generate well-defined path integrals and Green functions. Moreover, S will be invariant with respect to the action of a differential operator, s, called *BFV-BRST operator* or *BFV-BRST symmetry.* Any other observable F of the system will meet the same invariance condition:

$$sS = 0;\ sF = 0 \tag{10}$$

The price we have to pay is given by the extension of the space in which the new action is defined. The extended space is generated by the initial physical coordinates but also by new ghost type generators. As in this paper we shall mainly use a Hamiltonian approach, we shall refer to the extended space as *extended phase space.*

An important requirement for s is its nilpotency:

$$s^2 = 0 \tag{11}$$

This requirement express the feature of differential operator for s. A very useful representation of s, ensuring a symplectic structure for the extended space and of implementing the symmetry s by means of a canonical transformation:

$$s* \equiv [*, \Omega]$$

In the previous equation Ω is called the *BFV-BRST charge* and the bracket is a *generalized Poisson bracket* written in terms of the whole set of canonical variables (real and ghost types). The nilpotency of s combined with the Jacobi identity leads to the *master equation,* the main equation which allows to effectively obtain Ω:

$$[\Omega, \Omega] = 0 \tag{12}$$

Together with Ω, another important observable in the Hamiltonian approach is the Hamiltonian itself. As the action, it has to be "extended" in order to become a BRST invariant function:

$$H_0 \rightarrow H = H_0 + ...; sH \equiv [H, S] = 0$$

The extended Hamiltonian H and the extended action S still contain non-physical degrees of freedom coming from the gauge symmetry H_0 and S_0. They have to be killed by imposing adequate gauge fixing conditions. As a conclusion, the implementation of the BRST symmetry imposes the following algorithm: (i) *the construction of an adequate extended phase space* where the ghost type variables are added to the real ones; (ii) *the construction of the BFV-BRST charges and the extended Hamiltonian*; (iii) *the gauge fixing procedure.*

2.3 The sp(3) BRST Hamiltonian theory

We shall consider a gauge theory which at Hamiltonian level is represented by a constrained dynamical system described by the set of irreducible constraints $\{G_\alpha(q^i, p_i), \alpha = 1, \cdots, m,\ i = 1, \cdots, n\}$ and by the canonical Hamiltonian $H_0(q^i, p_i)$. The Grassmann parities of the constraints and of the Hamiltonian are $\varepsilon(G_\alpha) = \varepsilon_\alpha,\ \varepsilon(H_0) = 0$. The gauge algebra have the form

$$[G_\alpha, G_\beta] = f^\gamma_{\alpha\beta} G_\gamma,\ [H_0, G_\alpha] = V^\beta_\alpha G_\beta \tag{13}$$

where the structure functions $f^\gamma_{\alpha\beta}$ and V^β_α can depend in general on the q^i and p_i.

For this theory we shall develop the $sp(3)$ BRST Hamiltonian approach [**?**]. Hence, we shall pass from the original gauge symmetry to a global symmetry, $sp(3)$ BRST symmetry

$$s^T = s_1 + s_2 + s_3, \tag{14}$$

$$s_a = \delta_a + d_a + \cdots,\ a = 1, 2, 3 \tag{15}$$

$$s_a s_b + s_b s_a = 0,\ a, b = 1, 2, 3. \tag{16}$$

The main steps which need to be followed are: *(i) the construction of the extended phase space* adequate for implementation of the $sp(3)$ BRST symmetry (14)-(16); *(ii) the construction of the BFV-BRST charges and the extended Hamiltonian*; *(iii) the gauge fixing procedure.*

The extended phase space will be generate by the introduction, for each constraint $G_\alpha, \alpha = 1, \cdots, m$, of the three pairs of canonical conjugate ghost variables $\{P_{\alpha a}, Q^{\alpha a},\ a = 1, 2, 3\}$ with $\varepsilon(P_{\alpha a}) = \varepsilon(Q^{\alpha a}) = \varepsilon_\alpha + 1$ and

$$\delta_a P_{\alpha b} = \delta_{ab} G_\alpha \tag{17}$$

$$d_a Q^{\alpha b} = \varepsilon_{adc}\delta^{db}\lambda^{\alpha c} + \frac{1}{2} f^\alpha_{\beta\gamma} Q^\beta Q^\gamma. \tag{18}$$

In order to secure the crucial property of the Koszul differentials, $\delta_a,\ a = 1, 2, 3$ namely the acyclicity of the positive resolution numbers, it is necessary to introduce the new generators, $\pi_{\alpha a}$ with $\varepsilon(\pi_{\alpha a}) = \varepsilon_a$ and their conjugates$\lambda^{\alpha a}$ with $\varepsilon(\lambda^{\alpha a}) = \varepsilon(\pi_{\alpha a}) = \varepsilon_a$ so that

$$\delta_a \pi_{\alpha b} = \varepsilon_{abc} P_{\alpha c} \tag{19}$$

$$d_a \lambda^{\alpha b} = \delta^b_a \eta^\alpha + \frac{1}{2} f^\alpha_{\beta\gamma}\lambda^{\beta b} Q^{\gamma c}\delta_{ca} + \frac{1}{12} f^\theta_{\sigma\alpha} f^\gamma_{\theta\beta}\varepsilon_{bcd} Q^{\alpha c} Q^{\beta d} Q^{\sigma e}\delta_{ea}. \tag{20}$$

The same property, the acyclicity of δ_a, imposes the introduction of new generators, π_α, and their conjugate η^α with $\varepsilon(\pi_\alpha) = \varepsilon(\eta^\alpha) = \varepsilon_a + 1$ and

$$\delta_a \pi_\alpha = \delta_{ab}\pi_{\alpha b} \tag{21}$$

$$d_a \eta^\alpha = \frac{1}{2} f^\alpha_{\beta\gamma}\eta^\beta Q^{\gamma b}\delta_{ba} + \frac{1}{12}(f^\theta_{\sigma\gamma} f^\alpha_{\theta\beta} + f^\theta_{\sigma\beta} f^\alpha_{\theta\gamma})\lambda^{\gamma c} Q^{\sigma b}\delta_{ba} Q^{\beta c}. \tag{22}$$

We will denote the whole set of generators of the extended phase space by

$$Q^A = \{q^i, Q^{\alpha a}, \lambda^{\alpha a}, \eta^\alpha,\ a = 1, 2, 3\} \tag{23}$$

$$P_A = \{p_i, P_{\alpha a}, \pi_{\alpha a}, \pi_\alpha,\ a = 1, 2, 3\}. \tag{24}$$

For two arbitrary functionals F and G defined by the extended phase space, the generalized Poisson brackets with respect to which the conjugation is defined are

$$[F,G]=\frac{\delta F}{\delta Q^A}\frac{\delta G}{\delta P_A}-(-)^{\varepsilon_F\varepsilon_G}\frac{\delta G}{\delta Q^A}\frac{\delta F}{\delta P_A}. \tag{25}$$

The graduation rules of all generators and operators of our theory assume the introduction of the following degree: (i) *ghost number* (gh) which is positive for ghosts, $gh(Q^A)>0$, negative for ghost momenta, $gh(P_A)<0$, and zero for real fields, $gh(q^i)=0=gh(p_i)$; (ii) *resolution degree* (res) which is positive for ghost momenta, $res(P_A)=-gh(P_A)$, and zero for ghosts, $res(Q^A)=0$ and real fields, $res(q^i)=0=res(p_i)$; (iii) level number (lev) which is positive for ghosts, $lev(Q^A)\geq 0$, negative for ghost momenta, $lev(P_A)\leq 0$ and zero for real fields, $lev(q^i)=lev(p_i)=0$. For the operators we define the following graduation: $gh(\delta_a)=1$, $lev(\delta_a)=a-1$, $gh(d_a)=1$, $lev(d_a)=a-1$, $gh(s_a)=1$, $lev(s_a)=a-1$.

The main quantities of the theory are the three BRST charges and the extended Hamiltonian. The BRST charges represent the canonical transcription of the BRST symmetries:

$$s_a * = [*,\Omega_a],\ a=1,2,3. \tag{26}$$

The relation (16) imposes the fulfilling of the following equations:

$$[\Omega_a,\Omega_b]=0;\ a,b=1,2,3. \tag{27}$$

These equations must be completed by adequate boundary conditions:

$$\left.\frac{\delta\Omega_a}{\delta Q^{\alpha b}}\right|_{P_A=0}=\delta_{ab}G_\alpha,\ \left.\frac{\delta\Omega_a}{\delta\lambda^{\alpha b}}\right|_{Q^A=P'_A=0}=\varepsilon_{abc}P_{\alpha c},\ \left.\frac{\delta\Omega_a}{\delta\eta^\alpha}\right|_{Q^A=P'_A=0}=\delta_{ab}\pi_\alpha \tag{28}$$

where P'_A denotes all the ghost momenta excepting the one appearing in the right hand side of the same relation. For the extended Hamiltonian the problem is

$$[H,\Omega_a]=0\,;\ a=1,2,3. \tag{29}$$

$$H|_{P=\pi=Q=\lambda=0}=H_0. \tag{30}$$

The solutions to the problems (27), (28) and (29), (30) depend on the type of the theory we deal with. As we will study the Yang-Mills fields, which are bosonic fields and which define a first rank irreducible theory where the structure functions $f^\gamma_{\alpha\beta}$ and V^α_β are constants, the *BFV-BRST charges* have the form [4]:

$$\Omega_a=G_\alpha Q^{\alpha b}\delta_{ba}+\varepsilon_{abc}P_{\alpha c}\lambda^{\alpha b}+\frac{1}{2}f^\alpha_{\beta\gamma}P_{\alpha c}Q^{\gamma c}Q^{\beta a}+\pi_{\alpha a}\eta^\alpha+\frac{1}{2}f^\alpha_{\beta\gamma}\pi_{\alpha c}\lambda^{\gamma c}Q^{\beta a}+$$
$$+\frac{1}{12}f^\theta_{\sigma\alpha}f^\gamma_{\theta\beta}\varepsilon_{bcd}\pi_{\alpha b}Q^{\alpha c}Q^{\beta d}Q^{\sigma e}\delta_{ea}+\frac{1}{2}f^\alpha_{\beta\gamma}\pi_\alpha\eta^\gamma Q^{\beta a}+$$
$$+\frac{1}{12}(f^\theta_{\sigma\alpha}f^\gamma_{\theta\beta}+f^\theta_{\sigma\beta}f^\gamma_{\theta\alpha})\pi_\gamma\lambda^{\alpha c}Q^{\sigma b}\delta_{ba}Q^{\beta c}$$

The extended Hamiltonian [4] will be

$$H=H_0+V^\alpha_\beta(P_{\alpha a}Q^{\beta a}+\pi_{\alpha a}\lambda^{\beta a}+\pi_a\eta^\beta). \tag{31}$$

In order to avoid the presence of any non-physical degrees of freedom in the theory, we have to apply the gauge fixing procedure. Unfortunately, new problems could be generated by the fact that it is difficult to choose a particular form of the gauge fixing term so that the covariance of the theory will not affected. We shall propose a general term to overcome such problems and to contain as ghosts the π-*momenta* of zero order only, the last momenta ensuring the acyclicity of the Koszul differentials. More concretely, the following theorem [**?**] is valid:

Theorem: For any BRST invariant function K a non-constant odd function Y, defined on the extended phase space $sp(3)$ and with $gh(Y) = -3$ exists so that

$$K = \frac{1}{3!}\varepsilon^{abc}[\Omega_a, [\Omega_b, [\Omega_c, Y]]]. \tag{32}$$

For the phase space generated by (23) and (24) the gauge fixing function Y has the form (in the De Witt notations):

$$Y = f^{\alpha}(q,p)\pi_{\alpha}. \tag{33}$$

3 The Yang-Mills field theory

In this section, we shall discuss the Yang-Mills theory in the $sp(3)$ BRST Hamiltonian approach. We will shall from the action which describes the non-abelian Yang-Mills fields in four dimensions

$$S_0[A^m_\mu] = -\frac{1}{4}\int d^4x\; F^m_{\mu\nu}F^{\mu\nu}_m \tag{34}$$

where

$$F^m_{\mu\nu} = \partial_\mu A^m_\nu - \partial_\nu A^m_\mu + g\varepsilon^m_{nr}A^n_\mu A^r_\nu. \tag{35}$$

The canonical analysis of the model leads to the irreducible first class constraints

$$G^{(1)}_m(x) \equiv p^0_m(x) \approx 0 \tag{36}$$

$$G^{(2)}_m(x) \equiv -\partial_i p^i_m(x) + g\varepsilon_{mnr}A^r_i(x)p^{in}(x) \approx 0 \tag{37}$$

and to the first class Hamiltonian

$$H_0 = \int d^3x\,(\frac{1}{4}F^m_{ij}F^{ij}_m - \frac{1}{2}p_{im}p^{im} + A^m_0(-\partial^i p_{im} + g\varepsilon^r_{mn}\,A^{in}p_{ir})). \tag{38}$$

The gauge algebra is given by:

$$[G^{(1)}_m, G^{(1)}_n] = 0,\;\; [G^{(1)}_m, G^{(2)}_n] = 0,\;\; [G^{(2)}_m, G^{(2)}_n] = g\varepsilon^r_{mn}G^{(2)}_r,$$
$$[H_0, G^{(1)}_m] = G^{(2)}_m, \qquad [H_0, G^{(2)}_m] = -g\varepsilon^r_{mn}A^{0n}G^{(2)}_r. \tag{39}$$

We shall denote the whole set of constraints as $\{G^{(\Delta)}_m, \Delta = 1,2; m = 1,\cdots,d\}$.

The generators of the extended phase space have the form

$$P_A \equiv \{p_{im}, P^{(\Delta)}_{ma}, \pi^{(\Delta)}_{mab}, \pi^{(\Delta)}_{ma}, \pi^{(\Delta)}_m\}, Q^A \equiv \{A^{im}, Q^{(\Delta)ma}, \lambda^{(\Delta)mab}, \lambda^{(\Delta)ma}, \eta^{(\Delta)m}\} \tag{40}$$

The nontrivial generalized Poisson brackets are

$$[p_{im}(x), A^{jn}(y)]_{x^0=y^0} = -\delta^j_i\delta(\mathbf{x}-\mathbf{y}), \tag{41}$$

$$[P^{(\Delta)}_{ma}(x), Q^{(\Delta')nb}(y)]_{x^0=y^0} = -\delta^n_m\delta^b_a\delta^{\Delta\Delta'}\delta(\mathbf{x}-\mathbf{y}), \tag{42}$$

$$[\pi^{(\Delta)}_{ma}(x), \lambda^{(\Delta')nb}(y)]_{x^0=y^0} = -\delta^n_m\delta^b_a\delta^{\Delta\Delta'}\delta(\mathbf{x}-\mathbf{y}), \tag{43}$$

$$[\pi^{(\Delta)}_m(x), \eta^{(\Delta')n}(y)]_{x^0=y^0} = -\delta^n_m\delta^{\Delta\Delta'}\delta(\mathbf{x}-\mathbf{y}). \tag{44}$$

As it is easily to verify, the BFV-BRST charges, solutions of the equation (27), have the following expressions:

$$\Omega_a = \int d^3x\;\;(p_{0m}Q^{(1)mb}\delta_{ba} + (-\partial^i p_{im} + g\varepsilon^r_{mn}A^{in}p_{ir})Q^{(2)mb}\delta_{ba}+$$

$$+\varepsilon_{abc}P^{(\Delta)}_{mc}\lambda^{(\Delta)mb} + \pi^{(\Delta)}_{ma}\eta^{(\Delta)m} + \frac{1}{2}g\varepsilon^m_{nr}(P^{(2)}_{mc}Q^{(2)rc} + \pi^{(2)}_{mc}\lambda^{(2)rc} + \pi^{(2)}_m\eta^{(2)r})Q^{(2)nb}\delta_{ba}+$$

$$+\frac{1}{12}g^2\varepsilon^e_{rn}\varepsilon^m_{eq}\varepsilon_{dbc}\pi^{(2)}_{mb}Q^{(2)rc}Q^{(2)qd}Q^{(2)nb}\delta_{ba}+ \tag{45}$$

$$+\frac{1}{12}g^2(\varepsilon^e_{nr}\varepsilon^m_{eq}+\varepsilon^e_{nq}\varepsilon^m_{er})\pi^{(2)}_{m}Q^{(2)nb}\delta_{ba}Q^{(2)qc}\lambda^{(2)rc}),\, a=1,2,3$$

The extended Hamiltonian will be of the form:

$$H=H_0+\int d^3x\ ((Q^{(1)ma}+gA^{0n}\varepsilon^m_{nr}Q^{(2)ra})P^{(2)}_{ma}+(\lambda^{(1)ma}+gA^{0n}\varepsilon^m_{nr}\lambda^{(2)ra})\pi^{(2)}_{ma}+$$

$$+(\eta^{(1)m}+gA^{0n}\varepsilon^m_{nr}\eta^{(2)r})\pi^{(2)}_m+\frac{g}{2}\varepsilon^m_{nr}(\varepsilon_{abc}Q^{(2)nb}Q^{(1)rc}\pi^{(2)}_{ma}+(Q^{(1)na}\lambda^{(2)ra}-$$

$$-Q^{(2)na}\lambda^{(1)ra})\pi^{(2)}_m)+\frac{g^2}{6}\varepsilon^e_{qr}\varepsilon^m_{en}\varepsilon_{abc}Q^{(2)na}Q^{(2)rb}Q^{(1)qc}\pi^{(2)}_m). \tag{46}$$

For the gauge fixing procedure we shall choose the following form of the fermion function [**?**]

$$Y=\int d^3x\ (\partial^i A^m_i)\pi^{(1)}_m \tag{47}$$

which leads to the gauge fixed action

$$S_Y=\int d^4x(-\frac{1}{4}F^m_{\mu\nu}F^{\mu\nu}_m+p_{0m}(\partial^\mu A^m_\mu)+(\partial_\mu P^{(1)}_{ma})(D^\mu)^m_nQ^{(2)na}-$$

$$-(\partial_\mu\pi^{(1)}_{ma})(D^\mu)^m_n\lambda^{(2)na}+(\partial_\mu\pi^{(1)}_m)(D^\mu)^m_n\eta^{(2)n}+$$

$$+\frac{g}{2}\varepsilon^m_{nr}(\varepsilon_{abc}(\partial_\mu\pi^{(1)}_{ma})Q^{(2)nc}(D^\mu)^r_eQ^{(2)eb})+(\partial_\mu\pi^{(1)}_m)((D^\mu)^n_e\lambda^{(2)ea})Q^{(2)ra}-$$

$$-\lambda^{(2)na}(D^\mu)^r_eQ^{(2)ea}))+\frac{g^2}{6}\varepsilon^m_{ne}\varepsilon^e_{rq}\varepsilon_{abc}(\partial_\mu\pi^{(1)}_m)Q^{(2)qa}Q^{(2)nb}(D^\mu)^r_gQ^{(2)gc}) \tag{48}$$

4 Towards a generalized mechanical model

Let us consider now the Euler-Lagrange equations generated by the gauge fixed action (48) for the fields A^n_μ, Q^{na}, λ^{na}, η^n and their conjugate momentum. They will have the form:

$$\partial_\mu F^{\mu\nu}_m+g\varepsilon_{mnr}A^n_\mu F^{\mu\nu r}=0 \tag{49}$$

$$\partial_\mu(D^\mu)^m_nQ^{na}=0 \tag{50}$$

$$\partial_\mu(D^\mu)^m_n\lambda^{na}+\frac{g}{2}\varepsilon_{abc}\varepsilon^m_{nr}(\partial_\mu Q^{nc})(D^\mu)^r_eQ^{eb}=0 \tag{51}$$

$$\partial_\mu(D^\mu)^m_n\eta^n-\frac{g^2}{12}\varepsilon_{abc}\varepsilon^m_{ne}\varepsilon^e_{rq}Q^{nb}(\partial_\mu Q^{qa})(D^\mu)^r_eQ^{ec}+$$

$$+\frac{g^2}{2}\varepsilon^m_{er}\varepsilon^e_{nq}A^{\mu q}(\lambda^{na}(\partial_\mu Q^{ra})+(\partial_\mu\lambda^{na})Q^{ra})=0 \tag{52}$$

$$(D^\mu)^n_m\partial_\mu P_{na}-g\varepsilon_{abc}\varepsilon_{mnr}(\partial_\mu\pi_{nb})(D^\mu)^r_eQ^{ec}+\frac{g}{2}\varepsilon_{abc}\varepsilon_{mre}(D^\mu)^n_e\partial_\mu\pi_{na}Q^{rc}+$$

$$+g\varepsilon^n_{mr}(\partial_\mu\pi_n)(D^\mu)^r_e\lambda^{eb}\delta_{ba}+\frac{g}{2}\varepsilon^e_{mn}(D^\mu)^r_e\partial_\mu\pi_r\lambda^{nb}\delta_{ba}+$$

$$+\frac{g^2}{2}\varepsilon_{abc}\varepsilon^n_{re}\varepsilon^e_{mw}(\partial_\mu\pi_n)(D^\mu)^w_qQ^{qb}Q^{rc}+\frac{g^2}{6}\varepsilon_{abc}\varepsilon^w_{re}\varepsilon^e_{mq}(D^\mu)^m_w\partial_\mu\pi_mQ^{qb}Q^{rc}=0 \tag{53}$$

$$(D^\mu)^n_m\partial_\mu\pi_{na}-g\varepsilon_{mnr}(\partial_\mu\pi_n)(D^\mu)^r_eQ^{ea}=0 \tag{54}$$

$$(D^\mu)^n_m\partial_\mu\pi_n=0. \tag{55}$$

A first remark is that, on the basis of these equations, a large part of the terms corresponding to the ghost variables (containing ghosts) can be eliminated from the gauge fixed action. We shall not insist on this remark at this particular stage as it does not fall within the scope of this paper.

We are interested now in another topic, namely in transforming the Yang-Mills field in a "mechanical" model, a system with a finite number of degrees of freedom, where the fields are replaced as unknown variables by a set of color factors. For the gauge field , A_m^α, described by the equation (49), this approach has been done long ago [3]. It was expressed in terms of the color factors $f^{(m)}$ by introducing some orthogonal matrices $\mathcal{O}_m^i$. More precisely, the following hypotheses were considered:

$$A_m^0 = 0,\ \partial^j A_m^i = 0,\ A_m^i(t) = \frac{1}{g}\mathcal{O}_m^i f^{(m)}(t),\ \mathcal{O}_m^i \mathcal{O}_n^i = \delta_{mn} \tag{56}$$

Using (56) in (49) one obtains the system of "mechanical" equations:

$$\ddot{f}^{(m)} + f^{(m)}(\mathbf{f}^2 - f^{(m)2}) = 0. \tag{57}$$

This system has been intensively studied as a non-linear dynamical system and special periodic orbits, invariance or integrability cases have been pointed out [13].

What we propose now is the use of the same technique for the extended action S, i.e. for the whole system (49)-(55). In principle, by solving the equation (50) we obtain the solutions $Q^{ma}(\mathbf{r},t)$ which, introduced in (51), lead to the solutions for $\lambda^{ma}(\mathbf{r},t)$. Using the solutions for $Q^{ma}(\mathbf{r},t)$ and $\lambda^{ma}(\mathbf{r},t)$ in the equation (52), we can obtain the solutions for η^m $(\mathbf{r},t)$. Therefore, it is necessary to solve the equations (50). In this respect, we shall introduce a new set of color factors, $h^{(m)}$, and we shall choose, by similarity with (56), the following form for the fields $Q^{ma}(\mathbf{r},t)$:

$$Q^{ma}(\mathbf{r},t) = h^{(m)}(t)u^{ma}(\mathbf{r}) \tag{58}$$

It is important to note that it is not possible to express the ghost fields $Q^{ma}(\mathbf{r},t)$ using the same color factors $\{f^{(m)}, m = 1, ..., d\}$ as for the real fields A_m^i. In order that (58) represents a well-defined decomposition, we will choose the fields $u^{ma}(\mathbf{r})$ as a basis, so that

$$\partial_j u^{ma}(\mathbf{r}) = \varepsilon^{mnq}\mathcal{O}_j^n u^{qa}(\mathbf{r}) \tag{59}$$

The previous relation allows us to write:

$$\partial_j Q^{ma}(\mathbf{r},t) = \varepsilon^{mnq} h^{(m)}(t)\mathcal{O}_j^n u^{qa}(\mathbf{r}). \tag{60}$$

Using (56), (58) and (60) in (50) we have:

$$\ddot{h}^{(m)}(t) - 2h^{(m)}(t) + f^{(n)}h^{(q)} = 0,\ m \neq n \neq q,\ m, n, q = 1, \cdots, d. \tag{61}$$

Similarly, from (56) and (55) via

$$\pi_m(\mathbf{r},t) = h^{(m)}(t)w_m(\mathbf{r},t) \tag{62}$$

$$\partial_j w_m(\mathbf{r}) = \varepsilon_{mnq}\mathcal{O}_j^n w_q(\mathbf{r}) \tag{63}$$

we shall obtain the same equation (64). With the solutions of the form (62) in the equations (54) and (53) we can obtain the solutions of these equations, respectively $\pi_{ma}(\mathbf{r},t)$ and $P_{ma}(\mathbf{r},t)$ respectively.

Conclusion: The mechanical Yang-Mills model corresponding to the extended action S can be written in the form of a system of $2d$ equations with $2d$ unknown quantities $\{f^{(m)}, h^{(m)}, m = 1, ..., d\}$:

$$\begin{cases} \ddot{f}^{(m)}(t) + f^{(m)}(t)(\mathbf{f}^2(t) - f^{(m)2}(t)) + \sum\limits_{\substack{n,q \\ m\neq n\neq q}} h^{(n)}(t)h^{(q)}(t) = 0 \\ \ddot{h}^{(m)}(t) - 2h^{(m)}(t) + \sum\limits_{\substack{n,q \\ m\neq n\neq q}} f^{(n)}(t)h^{(q)}(t) = 0 \end{cases}. \tag{64}$$

In the particular case $m = 3$, the equations (64) generate a system of 6 differential equations with 6 unknown quantities $\{f^{(i)}, h^{(i)}, i = 1, 2, 3\}$:

$$\ddot{f}^{(1)}(t) + f^{(1)}(t)(f^{(2)2}(t) + f^{(3)2}(t)) + h^{(2)}(t)h^{(3)}(t) = 0 \quad (65)$$

$$\ddot{f}^{(2)}(t) + f^{(2)}(t)(f^{(1)2}(t) + f^{(3)2}(t)) + h^{(1)}(t)h^{(3)}(t) = 0 \quad (66)$$

$$\ddot{f}^{(3)}(t) + f^{(3)}(t)(f^{(1)2}(t) + f^{(2)2}(t)) + h^{(1)}(t)h^{(2)}(t) = 0 \quad (67)$$

$$\ddot{h}^{(1)}(t) - 2h^{(1)}(t) + f^{(2)}(t)h^{(3)}(t) + f^{(3)}(t)h^{(2)}(t) = 0 \quad (68)$$

$$\ddot{h}^{(2)}(t) - 2h^{(2)}(t) + f^{(3)}(t)h^{(1)}(t) + f^{(1)}(t)h^{(3)}(t) = 0 \quad (69)$$

$$\ddot{h}^{(3)}(t) - 2h^{(3)}(t) + f^{(1)}(t)h^{(2)}(t) + f^{(2)}(t)h^{(1)}(t) = 0 \quad (70)$$

By introduction of the notations:

$$f^{(1)} \equiv x, f^{(2)} \equiv y, f^{(3)} \equiv z, h^{(1)} \equiv u, h^{(2)} \equiv v, h^{(3)} \equiv w \quad (71)$$

these equations will take the form:

$$\begin{cases} \ddot{x} + x(y^2 + z^2) + vw = 0 \\ \ddot{y} + y(x^2 + z^2) + uw = 0 \\ \ddot{z} + z(x^2 + y^2) + uv = 0 \\ \ddot{u} - 2u + vz + wy = 0 \\ \ddot{v} - 2v + wx + uz = 0 \\ \ddot{w} - 2w + uy + vx = 0 \end{cases} \quad (72)$$

It is difficult to find out a general solution of this system or to extract interesting information on the system in this form. Instead, in order to see what analogy and influences can be established between the dynamics of the gauge fields and that of the ghost fields, we shall restrict ourselves to the study of a particular 4-dimensional case. Let us consider the case: $f^{(1)} \equiv x, f^{(2)} = f^{(3)} = y, h^{(1)} \equiv u, h^{(2)} = h^{(3)} \equiv v$. This case means in fact that two supplementary constraints have to be considered:

$$G_1 \equiv y - z = 0; \; G_2 \equiv v - w = 0 \quad (73)$$

By using the Dirac technique concerning the implementation of the constraints, with adequate Lagrange multipliers, the system (72) transforms in the following $2D$ differential system:

$$\begin{aligned} \ddot{x} + 2xy^2 + v^2 &= 0 \\ \ddot{y} + yx^2 + y^3 + uv &= 0 \\ \ddot{u} - 2u + 2vy &= 0 \\ \ddot{v} - 2v + vx + uy &= 0 \end{aligned} \quad (74)$$

Two types of evolutions could be studied for this type of mechanical system: (i) the evolution of each variable in time, (ii) the Lie symmetries of the system. Such studies performed for the free Mechanical Yang-Mills model provided some interesting periodical trajectories and proved the existence of an interesting class of point-like symmetries [14]. What is really important to note in our cases, (72) and (74), is that the equations for the real fields are coupled with the equations for the ghosts. The last ones seem to influence the dynamics of the former ones.

5 Conclusions

The paper focused on the time evolution of the variables which are currently used in the description of the Yang-Mills model, starting from the gauge fixed action. We showed how the Hamiltonian formalism, applied to the case when a $sp(3)$ BRST symmetry is implemented, can be developed. The main idea was to take into consideration not only the real non-abelian gauge fields but the ghosts fields too, i.e. to write the evolution equations for the whole set of fields which generated the extended phase space. After the study of the system in the frame of the Quantum Field Theory, we changed the context and transformed the system in a mechanical one. The fields were expressed in terms of two sets of color factors $\{f^{(i)}, h^{(i)}, i = 1, ..., D\}$and, consequently, the field equations became a system of $2D$ second order differential equations. The concrete form of these equations for $D = 3, D = 2$ was effectively written down. The main conclusions arising in the two different contexts in which the Yang-Mills theory was considered are the following: (i) as a gauge field theory in the BRST approach, the Yang-Mills model shows that part of the ghosts disappear in the asymptotic states because of their equations of motion. It is not possible to eliminate all the ghost fields, a vertex type interaction between the gauge fields and the ghosts is still present; (ii) as a nonlinear dynamical system, the extended mechanical Yang-Mills model mixes the evolutionary equations for the real non-abelian fields with those corresponding to the ghost type variables, that is the ghosts influence the dynamics of the real fields.

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