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Lifting and descending procedures for Lagrangians of higher order

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Abstract

The aim of the paper is to give a more simple form of the tangent space of order $k \geq 1$ of a manifold. We prove that this is diffeomorphic with the tangent of k^1–velocities and with the cotangent bundles of k^1–co-velocities. The diffeomorphism ic constructed using a totally reducible Lagrangian or a totally co-reducible Lagrangian of order k that can come, for example, from a hyperregular Lagrangian of first order on M.

Let M be a smooth manifold (all the objects considered in the paper are supposed to be of class C^∞). For every $k \in I\!N$ one can associate with M the differentiable manifolds T^kM, $T^{k*}M$, T^1_kM and $\left(T^1_k\right)^*$, in a functorial manner.

First, T^kM is the *tangent space of order* k, $T^0M = M$, $T^1M = TM$ (see [3, 5]). Then T^kM can be considered as a locally trivial bundle $T^kM \xrightarrow{\pi_j} T^jM$ for every $j = \overline{0, k-1}$. The dual counterpart of T^kM, as considered in [6, 10], is $T^{k*}M = T^{k-1}M \times_M T^*M$, the *cotangent space of order* r, where $\times_M$ denotes the fibered products of bundles over the base M. A *Lagrangian of order* k on M is $L : T^kM \to I\!R$; its dual counterpart, proposed in [10], is the *affine Hamiltonian* $h : T^kM^\dagger \to T^{k*}M$, a section of the affine one-dimensional affine bundle $T^kM^\dagger \xrightarrow{\Pi} T^{k*}M$, where $T^{k\dagger}M \to T^{k-1}M$ is the affine dual of the affine bundle $T^kM \xrightarrow{\pi_{k-1}} T^{k-1}M$. Hyperregular Lagrangians and affine Hamiltonians are naturally related by Legendre transformations.

The manifold T^1_kM is the Whitney sum $T^1_kM = \underbrace{TM \oplus \cdots \oplus TM}_{k \text{ times}}$; since it can be identified with the manifold $J^1_0(I\!R^k, M)$ of the k^1-velocities of M, it is called the *tangent bundle of* k^1*-velocities* of M (see [4, 8]). The dual bundle $\left(T^1_k\right)^* M = \underbrace{T^*M \oplus \cdots \oplus T^*M}_{k \text{ times}}$ is the vector bundle of k^1-*covelocities* of M (see also [4, 8]).

Two classes of high order Lagrangians of order k are considered: a *co-reducible* Lagrangian of order k that gives rise to a diffeomorphism of T^kM and T^{1*}_kM and a *reducible* Lagrangian of order k that gives rise to a diffeomorphism of T^kM and T^1_kM. A co-reducible Lagrangian induces a Hamiltonian $\tilde{H}$ on T^{1*}_kM and a reducible Lagrangian induces a Lagrangian $\tilde{L}$ on T^1_kM. If $\tilde{H}$ is hyperregular one say that L is co-hyperreducible and if $\tilde{L}$ is hyperregular one say that L is hyperreducible.

The lift of a hyperregular Lagrangian of first order to a Lagrangian of order k, constructed in Proposition 4), is co-hyperreducible and hyperreducible as well.

We use local coordinates from [5], but in spite of their local forms, the main objects are global ones.

A semispray of order k is a section $S : T^kM \to T^{k+1}M$ of the affine bundle $\pi_k : T^{k+1}M \to T^kM$. Since $T^{k+1}M \subset TT^kM$ (in fact π_k is an affine subbundle of the tangent bundle of T^kM), then S can be regarded as well as a vector field on T^kM.

Let us denote by $T^{k-1,1}M = T^{k-1}M \times_M TM$; more general, if $0 \leq r \leq k$, then $T^{r,k-r}M = T^rM \times_M T^1_{k-r}M$, where $T^0M = M = T^1_0M$.

Proposition 1 *If* $S : T^{k-1}M \to T^kM$ *is a semispray of order* k*, then there is a diffeomorphism* $\Phi : T^kM \to T^{k-1,1}M$*; more general, if* $0 \leq r \leq k$ *and* $S^{(\alpha)} : T^{\alpha-1}M \to T^\alpha M$*,* $\alpha = \overline{r+1, k}$ *are* *semisprays (of order α), then there is a diffeomorphism $\Phi^{(r)} : T^kM \to T^{r,k-r}M$. In particular, if $S^{(\alpha)} : T^{\alpha-1}M \to T^{\alpha}M$, $\alpha = \overline{2,k}$ are semisprays, then there is a diffeomorphism $\Phi^{(k)} : T^kM \to T^1_kM$.*

We say that a diffeomorphism $\Phi : T^kM \to T^1_kM$ is a *semi-spray type diffeomorphism* if it has the form $\Phi = \Phi^{(k)}$ as in Proposition above.

There is a semispray of order $k \geq 1$ canonically associated with a k-order Lagrangian L (see, for example, [5, 1]), given by a section $S : T^kM \to T^{k+1}M$, which in local coordinates has the form $(x^i, y^{(1)j}, \ldots, y^{(k)j}) \xrightarrow{S} (x^i, y^{(1)j}, \ldots, ky^{(k)j}, -(k+1)S^i(x^i, y^{(1)j}, \ldots, y^{(k)j}))$, where

$(k+1)S^i = \dfrac{1}{2}g^{ij}\left(d_T^{(k)}\left(\dfrac{\partial L}{\partial y^{(k)j}}\right) - \dfrac{\partial L}{\partial y^{(k-1)j}}\right)$ and

$d_T^{(k)} = y^{(1)i}\dfrac{\partial}{\partial x^i} + y^{(2)i}\dfrac{\partial}{\partial y^{(1)i}} + \cdots + y^{(k+1)i}\dfrac{\partial}{\partial y^{(k)i}}$ is the Tulczyjew operator, that has only a local action (it is not a global vector field).

Proposition 2 *Let $L : TM \to I\!R$ be a hyperregular Lagrangian (of first order). Then there is a semi-spray type diffeomorphism $\Phi : T^kM \to T^1_kM$ canonically associated with L.*

Notice that in particular the Lagrangian L may be a Finslerian if it is 2–homogeneous, or L may come from a Riemannian metric if it is quadratic in velocities.

If $\varepsilon_1, \ldots, \varepsilon_k$ are real numbers, $\varepsilon_i \neq 0$, $i \geq 1$, one can consider also a k-lagrangian $L^{(k)} : T^kM \to I\!R$ having the local form $L^{(k)}(x^i, y^{(1)i}, y^{(2)i}, \ldots, y^{(k)i}) = \varepsilon_1 L(x^i, y^{(1)i}) + \varepsilon_2 L(x^i, z^{(2)i}) + \cdots + \varepsilon_k L(x^i, z^{(k)i})$; it is a Lagrangian in the multisymplectic sense (see [3, 5]), using the coordinates $x^i, y^{(1)i}, z^{(2)i}, \ldots z^{(k)i})$ on T^kM. In general one can prove the following result.

Proposition 3 *Let $\{L_\alpha\}_{\alpha=\overline{1,k}}$, $L_\alpha : TM \to I\!R$ be hyperregular Lagrangians of first order $k \in I\!N^*$, $\alpha = \overline{1,k}$. Then there is a semi-spray type diffeomorphism $\Phi : T^kM \to T^1_kM$ canonically associated with $\{L_\alpha\}$.*

The diffeomorphism Φ can be constructed using Proposition 1, constructing inductively the Lagrangians $\{L^{(\alpha)}\}_{\alpha=\overline{1,k}}$ by formula $L^{(\alpha)}(x^i, y^{(1)i}, y^{(2)i}, \ldots, y^{(\alpha)i}) = L_1(x^i, y^{(1)i}) + L_2(x^i, z^{(2)i}) + \cdots + L_\alpha(x^i, z^{(\alpha)i})$, where succesively $z^{(\alpha)i}$ are constructed as in Proposition 2, using formula (**??**). □

According to [10], an *affine hamiltonian* of order k on M is a differentiable map $h : T^{k*}M \to \widetilde{T^kM^\dagger}$, such that $\Pi \circ h = 1_{\widetilde{T^{k*}M}}$, where $\Pi : \widetilde{T^kM^\dagger} \to \widetilde{T^{k*}M}$. Thus h has the local form $h(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) = (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i, -H_0(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i))$. The local functions H_0 change according to the rules $H'_0(x^{i'}, y^{(1)i'}, \ldots, y^{(k-1)i'}, p_{i'}) = H_0(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) + \dfrac{1}{k}\Gamma_U^{(k-1)}(y^{(k-1)i'})\dfrac{\partial x^i}{\partial x^{i'}}p_i$. It is easy to see that $\dfrac{\partial H'_0}{\partial p_{i'}} = \dfrac{\partial x^{i'}}{\partial x^i}\dfrac{\partial H_0}{\partial p_i} + \dfrac{1}{k}\Gamma_U^{(k-1)}(y^{(k-1)i'})$. Thus there is a map $\mathcal{H} : T^{k*}M \to T^kM$, given in local coordinates by $\mathcal{H}(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) = (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, \dfrac{\partial H_0}{\partial p_i}(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i))$, called the Legendre* mapping of the affine hamiltonian h. We say also that h is *regular* if $\mathcal{H}$ is a local diffeomorphism and h is *hiperregular* if $\mathcal{H}$ is a global diffeomorphism. Since $\dfrac{\partial^2 H'_0}{\partial p_{i'}\partial p_{j'}} = \dfrac{\partial x^{i'}}{\partial x^i}\dfrac{\partial x^{j'}}{\partial x^j}\dfrac{\partial^2 H_0}{\partial p_i \partial p_j}$, it follows that $h^{ij} = \dfrac{\partial^2 H_0}{\partial p_i \partial p_j}$ is a symmetric 2-contravariant d-tensor, which is non-degenerate iff h is regular. There is a real function $H : T^{k*}M \to I\!R$ defined by the formula $H(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) = p_i\dfrac{\partial H_0}{\partial p_i} - H_0$. We call H the *pseudo-energy* of h.

Let $L : T^kM \to I\!R$ be a hyperregular k-Lagrangian. The Legendre transformation $\mathcal{L} : T^kM \to T^{k*}M$ is a diffeomorphism and there is an affine Lagrangian h defined by L, using $\mathcal{L}$, as follows. Let $(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) \to (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, H^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i))$ be the local form of the inverse of $\mathcal{L}$. Then the local functions H_0 on $T^{k*}M$, defined by the formula

$$\begin{aligned} H_0(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) &= p_jH^j - \\ &\quad L(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, H^i), \end{aligned}$$

where $H^i = H^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i)$, define locally an affine hamiltonian of order k on M. Consider

$$\tilde{H}_0^{(k)} = \frac{\partial H_0}{\partial p_j} p_j - H_0, \tag{1}$$

which is a real function on $T^{k*}M$. Let us denote $\mathcal{L} = \mathcal{L}_{(k)}$ and define $\mathcal{L}_{(k-1)} : T^{k*}M \to T^{(k-1)*}M \times_M T^*M$ using the formula $\mathcal{L}_{(k-1)}(x^i,\ y^{(1)i},\ \ldots,\ y^{(k-1)i},\ p_i) = (x^i,\ y^{(1)i},\ \ldots,\ y^{(k-2)i},\ \frac{\partial \tilde{H}_0^{(k)}}{\partial y^{(k-1)i}},\ p_i)$. Let us denote $p_i = p_{(k)i}$ and $H_0 = H_0^{(k)}$ and let us suppose that $\mathcal{L}_{(k-1)}$ is a diffemorphism and let $\mathcal{L}_{(k-1)}^{-1}$ having the local form $(x^i,\ y^{(1)i},\ \ldots,\ y^{(k-2)i}, p_{(k-1)i},\ p_{(k)i}) \overset{\mathcal{L}_{(k-1)}^{-1}}{\to} (x^i,\ y^{(1)i},\ \ldots,\ y^{(k-2)i},\ H^i(x^i,\ y^{(1)i},\ \ldots,\ y^{(k-2)i},\ p_{(k-1)i},\ p_{(k)i}),\ p_{(k)i})$. Let

$$\begin{aligned} H_0^{(k-1)}(x^i, y^{(1)i}, \ldots, y^{(k-2)i}, p_{(k-1)i}, p_{(k)i}) \;&=\; p_{(k-1)j} H^i - \\ &\quad \tilde{H}_0^{(k)}(x^i, y^{(1)i}, \ldots, y^{(k-2)i}, H^i, p_{(k)i}), \end{aligned}$$

where $H^i = H^i(x^i, y^{(1)i}, \ldots, y^{(k-2)i}, p_{(k-1)i}, p_{(k)i})$

We consider in that follow a procedure that descends the degree of some higher order Hamiltonians, constructed using a given high order Lagrangian.

Consider $\tilde{H}_0^{(k-2)} = \frac{\partial H_0^{(k-1)}}{\partial p_{(k-1)j}} p_{(k-1)j} - H_0^{(k-1)}$, which is a real function on $T^{(k-1)*}M \times_M T^*M$. Inductively, let us suppose that the diffeomorphisms $\mathcal{L}_{(k)}, \ldots, \mathcal{L}_{(k-q)}$ has been constructed for $1 < q < k-1$. We have $\mathcal{L}_{(k-q)} : T^{k-q}M \times_M (T^*M)^q \to T^{(k-q)*}M \times_M (T^*M)^q = T^{(k-q-1)}M \times_M (T^*M)^{q+1}$, where $(T^*M)^q = T^*M \oplus \cdots \oplus T^*M$ (q times) is a diffemorphism, given by a formula $\mathcal{L}_{(k-q)}(x^i,\ y^{(1)i},\ \ldots,\ y^{(k-q)i}, p_{(k-q+1)i}, \ldots, p_{(k)i}) = (x^i, y^{(1)i}, \ldots, y^{(k-q-1)i},\ \frac{\partial \tilde{H}_0^{(q)}}{\partial y^{(k-q)i}}(x^i, y^{(1)i}, \ldots, y^{(k-q)i}, p_{(k-q+1)i}, \ldots, p_{(k)i}), p_{(k-q+1)i}, \ldots, p_{(k)i})$. Let $\mathcal{L}_{(k-q)}^{-1}$ having the local form $(x^i,\ y^{(1)i},\ \ldots,\ y^{(k-q-1)i}, p_{(k-q)i}, \ldots,\ p_{(k)i}) \overset{\mathcal{L}_{(k-q)}^{-1}}{\to} (x^i,\ y^{(1)i},\ \ldots,\ y^{(k-q-1)i},\ H^i(x^i,\ y^{(1)i},\ \ldots,\ y^{(k-q-1)i},\ p_{(k-q)i}, \ldots,\ p_{(k)i}),\ p_{(k-q+1)i}, \ldots,\ p_{(k)i})$. Let $H_0^{(k-q-1)}(x^i, y^{(1)i}, \ldots, y^{(k-q-2)i}, p_{(k-q-1)i}, \ldots, p_{(k)i}) = p_{(k-q-1)j}\ H^j(x^i, y^{(1)i}, \ldots, y^{(k-q-1)i}, p_{(k-q)i}, \ldots, p_{(k)i}) - \tilde{H}_0^{(k-q+1)}(x^i, y^{(1)i}, \ldots, y^{(k-q-1)i},\ H^i,$
$p_{(k-q+1)i},\ \ldots, p_{(k)i})$.

If $k-q-1 > 1$, we consider $\tilde{H}_0^{(k-q-1)} = \frac{\partial H_0^{(k-q-1)}}{\partial p_{(k-q-1)j}} p_{(k-q-1)j} - H_0^{(k-q-1)}$, which is a real function on $T^{(k-q-1)*}M \times_M (T^*M)^{q+1}$; we define $\mathcal{L}_{(k-q-1)} : T^{k-q-1}M \times_M (T^*M)^{q+1} \to T^{(k-q-1)*}M \times_M (T^*M)^{q+1} = T^{(k-q-2)}M \times_M (T^*M)^{q+2}$ using the formula $\mathcal{L}_{(k-q-1)}(x^i,\ y^{(1)i},\ \ldots,\ y^{(k-q-1)i},\ p_{(k-q)i}, \ldots, p_{(k)i}) = (x^i,\ y^{(1)i},\ \ldots,\ y^{(k-q-1)i},\ \frac{\partial \tilde{H}_0^{(k-q-1)}}{\partial y^{(k-q-1)i}}(x^i, y^{(1)i}, \ldots,\ y^{(k-q-1)i},\ p_{(k-q)i}, \ldots, p_{(k)i}),\ p_{(k-q+1)i}, \ldots, p_{(k)i})$. We suppose that $\mathcal{L}_{(k-q-1)}$ is a diffeomorphism.

If $k-q-1 = 1$, we skip $\tilde{H}_0^{(1)}$ and we consider $\mathcal{L}_{(1)} : TM \times_M (T^*M)^{k-1} \to T^*M \times_M (T^*M)^{k-1} = (T^*M)^k$ using the formula $\mathcal{L}_{(1)}(x^i, y^{(1)i}, p_{(2)i}, \ldots, p_{(k)i}) = (x^i,\ \frac{\partial H_0^{(1)}}{\partial y^{(1)i}}(x^i, y^{(1)i}, p_{(2)i}, \ldots, p_{(k)i}), p_{(2)i}, \ldots, p_{(k)i})$. We suppose also that $\mathcal{L}_{(1)}$ is a diffeomorphism and its inverse has the local form $\mathcal{L}_{(1)}(p_{(1)i}, \ldots, p_{(k)i}) = (H^i(p_{(1)i}, \ldots, p_{(k)i}), p_{(2)i}, \ldots, p_{(k)i})$. Let us define the multi-Hamiltonian $\tilde{H}^{(0)} : (T^*M)^k \to I\!R$ using the formula $\tilde{H}^{(0)}(p_{(1)i}, \ldots, p_{(k)i}) = p_{(1)i} H^i(p_{(1)i}, \ldots,\ p_{(k)i}) -\ H_0^{(1)}(H^i, p_{(2)i}, \ldots, p_{(k)i})$.

If we suppose that all the applications $\mathcal{L}_{(k)}, \ldots, \mathcal{L}_1$ are diffeomorphisms, we say that the Lagrangian L of order k is *co-reducible* Let us denote by $\Psi = \mathcal{L}_{(1)} \circ \cdots \circ \mathcal{L}_{(k)}$.

Theorem 1 *If the Lagrangian L of order $k \geq 1$ is co-reducible, then there is a diffeomorphism* $T^kM \overset{\Psi}{\to} \underbrace{TM^* \times_M \cdots \times_M TM^*}_{k\ times} = T_k^{1*}M$ *with the canonical k-cotangent structure on M such that* $L = \tilde{H}^{(0)} \circ \Psi$.

We prove below that always there is a completely regular Lagrangian of any order $k \geq 1$, constructed by lifting an arbitrary hyperregular Lagrangian of first order to order k.

Proposition 4 *Let $L : TM \to I\!R$ be a hyperregular Lagrangian and $L^{(k)} : T^kM \to I\!R$ be the Lagrangian constructed above, given by $L^{(k)}(x^i, y^{(1)i}, \ldots, y^{(k)i}) = L(x^i, y^{(1)i}) + L(x^i, z^{(2)i}) + \cdots + L(x^i, z^{(k)i})$. Then $L^{(k)}$ is a co-reducible Lagrangian of order k.*

Let us denote by $VT^kM \to T^kM$ the vertical vector bundle of the bundle $T^kM \overset{\pi_k}{\to} M$. A *non-linear connection* in the bundle π_k is defined by a vector subbundle $HT^kM \subset TT^kM$ that gives a Whitney sum $TT^kM = VT^kM \oplus HT^kM$. It follows a splitting $VT^kM = V^{k,1}M \oplus \cdots \oplus V^{k,k}M$, where $V^{k,\alpha}M = J^\alpha(HT^kM)$, $\alpha = \overline{1,k}$ and J is the k–tangent structure on T^kM (it has the local form $\frac{\partial}{\partial y^{(\alpha)i}} \overset{J}{\to} \frac{\partial}{\partial y^{(\alpha+1)i}}$, $\alpha = \overline{0, k-1}$, where $\frac{\partial}{\partial y^{(0)i}} = \frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial y^{(k)i}} \overset{J}{\to} 0$). See [5] for more details concerning these constructions.

The vector bundles $V^{k,\alpha}M$ are all isomorphic with HT^kM and also with the induced vector bundle π_k^*TM. There are canonical isomorphisms of the vector bundle $V^{k,k}M$ with the vector bundles $V^{k,\alpha}M$, $\alpha = \overline{1, k-1}$ and with HT^kM.

Using local coordinates, the isomorphisms of $V^{k,k}M$ with $V^{k,\alpha}M$, $\alpha = \overline{1, k-1}$ and with HT^kM have the local form $\frac{\partial}{\partial y^{(k)j}} \to \frac{\delta}{\delta y^{(\alpha)j}}$ and $\frac{\partial}{\partial y^{(k)j}} \to \frac{\delta}{\delta x^j}$ respectively, where $\frac{\delta}{\delta y^{(0)i}} = \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \underset{1}{N}{}_i^j \frac{\partial}{\partial y^{(1)j}} - \cdots - \underset{k}{N}{}_i^j \frac{\partial}{\partial y^{(k)j}}$, $\frac{\delta}{\delta y^{(\alpha)j}} = \frac{\partial}{\partial y^{(\alpha)j}} - \underset{1}{N}{}_i^j \frac{\partial}{\partial y^{(\alpha+1)j}} - \cdots - \underset{k-\alpha}{N}{}_i^j \frac{\partial}{\partial y^{(k)j}}$. The local functions $\underset{1}{N}{}_i^j, \ldots, \underset{k}{N}{}_i^j$ depend on $(x^i, y^{(1)i}, \ldots, y^{(k)i})$.

The embedding $T^{k+1}M \overset{I_{k+1}}{\to} TT^kM$ has the local form $(x^i, y^{(1)i}, \ldots, y^{(k+1)i}) \overset{I_{k+1}}{\to} y^{(1)i}\frac{\partial}{\partial x^i} + \cdots + (k+1)y^{(k+1)i}\frac{\partial}{\partial y^{(k)i}} = \tilde{y}^{(1)i}\frac{\delta}{\delta x^i} + \cdots + (k+1)\tilde{y}^{(k+1)i}\frac{\delta}{\delta y^{(k)i}}$, where

$$
\begin{aligned}
&y^{(1)i} = \tilde{y}^{(1)i},\\
&2y^{(2)i} = 2\tilde{y}^{(2)i} - \underset{1}{N}{}_j^i \, \tilde{y}^{(1)j},\\
&\ldots\\
&ky^{(k)i} = k\tilde{y}^{(k)i} - \underset{1}{N}{}_j^i \, \tilde{y}^{(k-1)j} - \cdots - \underset{k-1}{N}{}_j^i \, \tilde{y}^{(1)j}\\
&(k+1)y^{(k+1)i} = (k+1)\tilde{y}^{(k+1)i} - \underset{1}{N}{}_j^i \, \tilde{y}^{(k)j} - \cdots - \underset{k}{N}{}_j^i \, \tilde{y}^{(1)j}.
\end{aligned}
$$

It is easy to see that the applications $(x^i, y^{(1)i}, \ldots, y^{(k)i}) \overset{\Lambda}{\to} (x^i, \tilde{y}^{(1)i}, \ldots, \tilde{y}^{(k)i})$ and $(x^i, y^{(1)i}, \ldots, y^{(k+1)i}) \overset{\Lambda'}{\to} (x^i, \tilde{y}^{(1)i}, \ldots, \tilde{y}^{(k+1)i})$ gives some differentiable maps $\Lambda : T^kM \to T^1_kM$ and $\Lambda' : T^{k+1}M \to T^1_{k+1}M$. They depend only on the non-linear connection in the bundle $\pi_k : T^kM \to M$.

Proposition 5 *A non-linear connection in the bundle $\pi_k : T^kM \to M$ defines in a canonical way some differentiable maps $\Lambda : T^kM \to T^1_kM$ and $\Lambda' : T^{k+1}M \to T^1_{k+1}M$.*

We say that a non-linear connection in the bundle $\pi_k : T^kM \to M$ is *regular* (*hyperregular*) if the map Λ is a local diffeomorphism (global diffeomorphism). It is easy to see that in this case Λ' is also a diffeomorphism.

Examples of hyperregular non-linear connections can be constructed as follows. Let us say that a non-linear connection of order $k \geq 2$ is *totally projectable* if the horizontal bundle $HT^kM = H^{(k)}T^kM \subset TT^kM$ projects, according to the succeeding differentials of the canonical projections $T^kM \overset{\Pi_k}{\to} T^{k-1}M \overset{\Pi_{k-1}}{\to} \cdots \overset{\Pi_2}{\to} T^1M$, to horizontal distributions $H^{(\alpha)}T^\alpha M \subset TT^\alpha M$ that give non-linear connections of order $\alpha = \overline{1, k-1}$. The local coefficients $\{N_j^{(\alpha)i}\}_{\alpha=\overline{1,k}}$ of the given non-linear connection have the property in this case that $N_j^{(\alpha)i} = N_j^{(\alpha)i}(x^l, y^{(1)l}, \ldots, , y^{(\alpha)l})$, $\alpha = \overline{1,k}$, and $\{N_j^{(\beta)i}\}_{\beta=\overline{1,\alpha}}$ are the coefficients of the suitable non-linear connection of order α. Taking into account the local form of the map Λ, it is easy to see that the following statement is true.

Proposition 6 *A totally projectable non-linear connection of order $k \geq 2$ is a hyperregular one.*

Let g be a Riemannian metric on the manifold $T^k M$. Then the vector subbundle $HT^k M = (VT^k M)^\perp \subset TT^k M$ defines a non-linear connection in the bundle π_k. We can consider a new Riemannian metric $\tilde{g}$ on the manifold $T^k M$ obtained from the restriction of g to $V^{k,\alpha}M$, then asking that the vector bundles $V^{k,\alpha}M$, $\alpha = \overline{1,k}$ and $HT^k M$ be orthogonal each to the other according to the metric $\tilde{g}$.

Using local coordinates, let us denote by $g_{ij}^{(\alpha)}(x^i, y^{(1)i}, \ldots, y^{(k)i}) = g\left(\frac{\partial}{\partial y^{(\alpha)i}}, \frac{\partial}{\partial y^{(\alpha)j}}\right)$ the local components of the metric g when it is resticted to the vector bundle $V^{k,\alpha}M$. Then $g_{ij}^{(\alpha)} = \tilde{g}\left(\frac{\delta}{\delta y^{(\alpha)i}}, \frac{\delta}{\delta y^{(\alpha)j}}\right)$, $\alpha = \overline{0,k}$, $\tilde{g}\left(\frac{\delta}{\delta y^{(\alpha)i}}, \frac{\delta}{\delta y^{(\beta)j}}\right) = 0$, $\alpha \neq \beta \in \{0, \ldots, k\}$.

Using Proposition 5 in the case of the non-linear connection constructed using the Riemannian metric g or $\tilde{g}$, one obtain a diffeomorphism Λ of $T^{k+1}M$ and $T^1_{k+1}M$. Using this diffeomorphism, one obtain a Riemannian metric on $T^1_{k+1}M$ that makes Λ an isometry.

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